Spanning 2-Trails from Degree Sum Conditions

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Abstract: Suppose G is a simple connected n-vertex graph. Let \( \sigma_3(G) \) denote the minimum degree sum of three independent vertices in G (which is \( \infty \) if G has no set of three independent vertices). A 2-trail is a trail that uses every vertex at most twice. Spanning 2-trails generalize hamilton paths and cycles. We prove three main results. First, if \( \sigma_3(G) \geq n - 1 \), then G has a spanning 2-trail, unless \( G \cong K_{1,3} \). Second, if \( \sigma_3(G) \geq n \), then G has either a hamilton path or a closed spanning 2-trail. Third, if G is 2-edge-connected and \( \sigma_3(G) \geq n \), then G has a closed spanning 2-trail, unless \( G \cong K_{2,3} \) or \( K_{2,3}^* \) (the 6-vertex graph obtained from \( K_{2,3} \) by subdividing one edge). All three results are sharp. These results are related to the study of connected and 2-edge-connected factors, spanning k-walks, even factors,

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and supereulerian graphs. In particular, a closed spanning 2-trail may be regarded as a connected (and 2-edge-connected) even \([2,4]\)-factor.

Keywords: degree sum; spanning trail; connected even factor

1. INTRODUCTION

We use Bondy and Murty’s book [2] for terminology and notation not defined here, and consider finite simple graphs only.

A walk in a graph will be called a \(k\)-walk if each vertex is used no more than \(k\) times (if the walk is closed, so that the same vertex occurs as both first and last vertex, we count this as just one use of that vertex). A \(k\)-trail is a \(k\)-walk with no repeated edges. For the purposes of the current paper, a \(k\)-walk or \(k\)-trail need not be closed, and need not be spanning. Note that a closed spanning 1-walk or 1-trail is a hamilton cycle, and an open spanning 1-walk or 1-trail is a hamilton path. A graph with a closed spanning trail has a spanning Eulerian subgraph, and is sometimes called supereulerian.

The degree of a vertex \(v\) in the graph \(G\) will be denoted \(\deg_G(v)\), or just \(\deg(v)\). The set of vertices adjacent to \(v\) in \(G\) will be denoted \(N_G(v)\) or just \(N(v)\). The degree sum of independent sets of vertices of given cardinality has often been used to give sufficient conditions for the existence of hamilton paths or cycles, or other structures such as spanning trails. The following notation will be useful:

\[
\sigma_k(G) = \min\{\Sigma_{i=1}^k \deg(v_i) \mid \{v_1, v_2, \ldots, v_k\} \text{ is independent in } G\}.
\]

If there are no independent sets of cardinality \(k\), \(\sigma_k(G)\) is taken to be \(\infty\). Our goal in this paper is to give conditions involving \(\sigma_3\) for spanning 2-trails.

Spanning \(k\)-trails are related to many other types of spanning subgraphs. Some relationships are shown in Figure 1: a closed spanning \(k\)-trail is a generalization of a hamilton cycle, and a special case of several other types of spanning subgraphs that have been examined previously. In particular, the existence of a closed spanning \(k\)-trail is equivalent to the existence of a connected (or 2-edge-connected) even \([2, 2k]\)-factor. It is also equivalent to the existence of a connected covering of the vertices of the graph by edge-disjoint cycles with each vertex appearing in at most \(k\) of the cycles. Spanning \(k\)-trails therefore seem to be a natural subject for investigation.

Degree sum results (involving \(\sigma_k\), for some \(k\)) are common as sufficient conditions for the existence of spanning subgraphs obeying degree and connectivity restrictions. The first degree sum result was a condition for hamilton cycles by Ore, generalizing a well known result of Dirac based on the minimum degree.

**Theorem 1.1** (Ore [17]). If \(G\) is an \(n\)-vertex graph with \(n \geq 3\) and \(\sigma_2(G) \geq n\), then \(G\) has a hamilton cycle.
Corollary 1.1. If $G$ is an $n$-vertex graph with $\sigma_2(G) \geq n - 1$, then $G$ has a hamilton path.

A result on $k$-walks that generalizes Theorem 1.1 was found by Jackson and Wormald.

Theorem 1.2 (Jackson and Wormald [12]). If $G$ is a connected $n$-vertex graph with $n \geq 3$ and $\sigma_{k+1}(G) \geq n$, where $k \geq 1$, then $G$ has a closed spanning $k$-walk.

For trails, which are the subject of this article, there are several results involving $\sigma_k$ that guarantee the existence of a spanning trail. However, none of these give a bound on how often each vertex is used. We begin with two results for $\sigma_2$.

Theorem 1.3 (Lesniak-Foster and Williamson [15]). If $G$ is an $n$-vertex graph with $n \geq 3$, $\delta(G) \geq 2$, and $\sigma_2(G) \geq n - 1$, then $G$ has a closed spanning trail.

Since a graph with $\sigma_2 \geq n - 1$ has $\delta \geq 2$ if and only if it is 2-edge-connected, the following is a strengthening of Theorem 1.3.

Theorem 1.4 (Benhocine et al. [1]). If $G$ is a 2-edge-connected $n$-vertex graph with $n \geq 3$ and $\sigma_2(G) \geq (2n + 3)/3$, then $G$ has a closed spanning trail.

These results have been improved further by Catlin [3] and Chen [5], provided $n$ is taken to be sufficiently large.

The above results deal with closed trails. If the trails may be open, they are easier to find. Lesniak-Foster and Williamson [15] mention that if $G$ is a connected $n$-vertex graph with $n \geq 5$ and $\sigma_2(G) \geq n - 2$, then $G$ contains a (possibly open) spanning trail. The following result involving $\sigma_3$ strengthens this.

Theorem 1.5 (Veldman [19]). If $G$ is a connected $n$-vertex graph with $n \geq 5$ and $\sigma_3(G) \geq n - 1$, then $G$ has a (possibly open) spanning trail.

*FIGURE 1. Hierarchy of spanning subgraphs.*
This has been further extended, as follows.

**Theorem 1.6** (Chen and Xue, cited in [6]). If $G$ is a connected $n$-vertex graph with $\sigma_3(G) \geq n - 2$, then either $G$ has a spanning trail (possibly open), or $G$ is isomorphic to one of $K_{1,4}, C_4^{++}$ (the 6-vertex graph obtained by attaching pendant edges to opposite vertices of a 4-cycle), or $A_n$ with $n \geq 4$ (see the end of Section 2).

Catlin [4] showed that some stronger conclusions could be reached if $\sigma_3 \geq n + 1$.

Our results in this article strengthen Theorems 1.4 and 1.5, and we obtain results for 2-trails that parallel Theorem 1.1 and Corollary 1.1. Our main theorems, Theorems 2.1, 3.1 and 4.1, are all sharp. With the exception of results by Gao and Wormald [11] for triangulations of the plane, projective plane, torus and Klein bottle, these are the first results that we are aware of that deal with spanning $k$-trails for $k \geq 2$.

As mentioned earlier, spanning $k$-trails are related to connected factors with degree conditions, and also to coverings of the vertices of a graph by cycles. Recent conditions based on degree sums for the existence of connected factors with degree restrictions include conditions involving $\sigma_2$ by Kouider and Mahéo [14] for 2-edge-connected $[2, b]$-factors, and by Nam [16] for connected $[a, b]$-factors. Conditions based on degree sums for coverings of the vertices of a graph by cycles, edges and vertices include work by Enomoto et al. [10] which was extended by Kouider and Lonc [13] and Saito [18]. Our results on closed 2-trails, Theorems 3.1 and 4.1, can be interpreted as results on the existence of connected even $[2, 4]$-factors, or as results on the existence of connected coverings of the vertices of a graph by edge-disjoint cycles with each vertex in at most 2 of the cycles.

## 2. OPEN 2-TRAILS

We begin by examining the existence of open spanning 2-trails in graphs. The following assumption will be used heavily in this section, and also in Section 3.

**Assumption 2.1.** Suppose $G$ is a connected $n$-vertex graph, and let $T = u_1u_2 \cdots u_p$ be a 2-trail in $G$ that has the minimum number of edges among all 2-trails covering the maximum number of vertices.

**Lemma 2.1.** Suppose Assumption 2.1 holds. Then $T$ is open, i.e., $u_1 \neq u_p$, and its ends $u_1$ and $u_p$ are used only once.

**Proof.** Suppose that $u_1 = u_j$ for some $j \geq 2$ (possibly $j = p$). Then $u_2u_3 \cdots u_p$ is a 2-trail covering the same vertices as $T$ but with fewer edges, contradicting Assumption 2.1. Thus, $u_1 \neq u_p$, and $u_1$ is used only once. Similarly, $u_p$ is used only once. 

\[\square\]
Lemma 2.2. Suppose Assumption 2.1 holds. If \( u \in V(T) \) is the initial vertex of an open 2-trail that covers \( V(T) \), then \( N(u) \subseteq V(T) \). In particular, \( N(u_1) \subseteq V(T) \) and \( N(u_p) \subseteq V(T) \).

Proof. Suppose there exists \( w \in N(u) \setminus V(T) \). Let \( (u = v_1)v_2v_3 \cdots v_q \) be the open 2-trail starting at \( u \) and covering \( V(T) \). Since \( T \) has the maximum number of vertices, \( v_1, v_2, \ldots, v_q \in V(T) \). Then \( wv_1v_2 \cdots v_q \) is an open 2-trail covering more vertices than \( T \), contradicting Assumption 2.1. ■

Define

\[
N^-(u_1) = \{u_{i-1} \mid u_1u_i \in E(G)\}, \\
N^+(u_p) = \{u_{i+1} \mid u_pu_i \in E(G)\}.
\]

These are both well-defined since neither \( u_1 \) nor \( u_p \) is adjacent to itself. For any vertex \( v \) and \( i = 1 \) or \( 2 \), let \( N_i(v) \) be the set of neighbours of \( v \) that are used \( i \) times on \( T \).

Lemma 2.3. Suppose Assumption 2.1 holds. If \( u_k \in N^-(u_1) \) or \( N^+(u_p) \) then \( u_k \) is used only once on \( T \), and \( N(u_k) \subseteq V(T) \).

Proof. If \( u_k \in N^-(u_1) \), then \( T' = u_ku_{k-1} \cdots u_1u_{k+1} \cdots u_p \) is a 2-trail from \( u_k \) to \( u_p \) with the same number of vertices and edges as \( T \). Therefore, \( T' \) satisfies Assumption 2.1. By Lemma 2.1, \( u_k \) is used only once on \( T' \) and hence only once on \( T \), and by Lemma 2.2, \( N(u_k) \subseteq V(T') = V(T) \). The argument when \( u_k \in N^+(u_p) \) is similar. ■

Lemma 2.4. Suppose Assumption 2.1 holds. Then \( |N^-(u_1)| = |N(u_1)| + |N_2(u_1)| \), and \( |N^+(u_p)| = |N(u_p)| + |N_2(u_p)| \).

Proof. \( |N(u_1)| + |N_2(u_1)| \) is the number of indices \( i \) for which \( u_i \in N(u_1) \). For each such \( i \), \( u_{i-1} \in N^-(u_1) \), and by Lemma 2.3 this counts every vertex in \( N^-(u_1) \) exactly once. The proof for \( |N^+(u_p)| \) is similar. ■

The main result of this section, which strengthens Theorem 1.5 by adding the restriction that the spanning trail uses every vertex at most twice, is as follows.

Theorem 2.1. Let \( G \) be a connected \( n \)-vertex graph with \( \sigma_3(G) \geq n - 1 \). Then either \( G \) has a (possibly open) spanning 2-trail, or \( G \cong K_{1,3} \).

Proof. Suppose \( G \) does not have a spanning 2-trail. Let \( T \) satisfy Assumption 2.1. Since \( G \) is connected, there exists a vertex \( w \in V(G) \setminus V(T) \) adjacent to a vertex of \( T \).

Claim 2.1A. There is no closed 2-trail \( T' \) with \( V(T') = V(T) \). In particular, \( u_1 \) and \( u_p \) are nonadjacent.

Proof. Write \( T' = v_1v_2 \cdots v_q \), with \( v_q = v_1 \). Since \( V(T') = V(T) \), \( w \) is adjacent to some \( v_i, i \leq q - 1 \). Then \( v_{i+1}v_{i+2} \cdots v_{q-1}(v_q = v_1)v_2 \cdots v_iw \) is a 2-trail covering more vertices than \( T' \) and \( T \), contradicting Assumption 2.1. ■
By Lemmas 2.1 and 2.2 and Claim 2.1A, $u_1, u_p$ and $w$ are distinct nonadjacent vertices, so $\deg(w) + \deg(u_1) + \deg(u_p) \geq n - 1$. Let $C = N^-(u_1) \cup N^+(u_p) \cup N(w)$.

Claim 2.1B. If $u_i \in N^-(u_1), u_j \in N^+(u_p), i < j$, and either $i \neq 1$ or $j \neq p$, then there exists $k$ with $i < k < j$ for which at least one of the following holds:

(i) $u_k$ is used only once on $T$, and $u_k \in V(T) \setminus C$, or (ii) $u_k \in N_2(u_1)$ and $u_k$ is the second occurrence of this vertex on $T$, or (iii) $u_k \in N_2(u_p)$ and $u_k$ is the first occurrence of this vertex on $T$.

Proof. We use induction on $j - i = d$.
Suppose $d = 1$. If $i = 1$ or $j = p$, then $u_1 u_p \in E(G)$. If $i > 1$ and $j < p$, then $u_1 u_j u_{i+1} \cdots u_p u_i u_{l-1} \cdots u_1$ is a closed 2-trail. In either case, Claim 2.1A is contradicted. So $d$ cannot be 1, and the result holds vacuously in this case.

Now assume the result holds when $j - i < d$, and consider the case where $j - i = d \geq 2$. Since $i \neq 1$ or $j \neq p$, we may assume without loss of generality that $i \neq 1$.

Suppose $u_{i+1}$ is used once on $T$. If $u_{i+1} \in N^-(u_1)$ or $N^+(u_p)$ then we may use induction. If $u_{i+1} \in N(w)$ then since $i + 1 \neq 2, wu_{i+1} u_1 u_2 \cdots u_p$ is a 2-trail covering more vertices than $T$, contradicting Assumption 2.1. So, $u_{i+1} \notin C$, and (i) holds.

Suppose $u_{i+1}$ is used twice on $T$. Then $u_{i+1} \in N_2(u_1)$. If $d = 2$, then $u_{i+1} = u_{j-1} \in N(u_p)$, and either (ii) or (iii) holds depending on whether $u_{i+1}$ is the second or first occurrence of this vertex, respectively. So, assume that $d \geq 3$. If $u_{i+1}$ is being used for the second time, then (ii) holds, so suppose $u_{i+1} = u_l$ where $i + 1 < l$. Then $T' = u_{i+2} u_{i+3} \cdots (u_l = u_{i+1}) u_1 u_2 \cdots (u_{i+1} = u_l) u_{i+2} \cdots u_p$ is a 2-trail covering $V(T)$ with the same number of edges as $T$, so Assumption 2.1 holds for $T'$. By Lemma 2.1, $u_{i+2}$ is used only once on $T'$, and hence only once on $T$. By Lemma 2.2, $N(u_{i+2}) \subseteq V(T') = V(T)$, so $u_{i+2} \notin N(w)$, and if $u_{i+2} \in N^-(u_1)$ or $N^+(u_p)$ we may use induction. Therefore, we may assume that $u_{i+2} \notin C$, and (i) holds.

Claim 2.1C. If either $\deg(u_1) \geq 2$ or $\deg(u_p) \geq 2$, then $|N_2(u_1)| + |N_2(u_p)| + |V(T) \setminus C| \geq |N^-(u_1) \cap N^+(u_p)| + 1$.

Proof. Write $N^-(u_1) \cap N^+(u_p) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_m}\}$, where $1 < i_1 < i_2 < \cdots < i_m < p$ and $m = |N^-(u_1) \cap N^+(u_p)|$. If $m = 1$, then applying Claim 2.1B to consecutive pairs of vertices in the sequence $u_1, u_{i_1}, u_{i_2}, \ldots, u_{i_m}, u_p$ we find $m + 1$ distinct indices $k$ satisfying (i), (ii) or (iii) of Claim 2.1B, of which there are at most $|V(T) \setminus C|, |N_2(u_1)|$, and $|N_2(u_p)|$, respectively.

If $m = 0$ and $\deg(u_1) \geq 2$, we apply Claim 2.1B to the last vertex $u_l$ of $N^-(u_1)$ on $T$ and $u_j = u_p$, and we use a similar argument if $\deg(u_p) \geq 2$.

Now we may complete the proof of Theorem 2.1. Suppose $\deg(u_1) = 1$ and $\deg(u_p) = 1$. Then $\deg(w) \geq n - 3$, but $w$ is not adjacent to $u_1, u_p$ or itself, so
deg(w) = n - 3, and w is adjacent to all other vertices of G. Since w has a
neighbor on T, p \geq 3. If p \geq 4, then w is adjacent to u_2 and u_3, and
u_1u_2wu_3u_4 \cdots u_p is a 2-trail with more vertices than T, a contradiction. Therefore
p = 3 and N(w) \cap V(T) = \{u_2\}. If w has a neighbor x not on T, then u_1u_2wx is a
2-trail with more vertices than T, so N(w) = \{u_2\}. Hence, 1 = \deg(w) = n - 3,
n = 4, and \( G \cong K_{1,3} \).

Now suppose that \deg(u_1) \geq 2 or \deg(u_p) \geq 2. Then
\[ |C| = |N(w)| + |N^-(u_1) \cup N^+(u_p)| \quad \text{by Lemma 2.3} \]
\[ = |N(w)| + |N^-(u_1)| + |N^+(u_p)| - |N^-(u_1) \cap N^+(u_p)| \]
\[ = |N(w)| + |N(u_1)| + |N_2(u_1)| + |N(u_p)| + |N_2(u_p)| - |N^-(u_1) \cap N^+(u_p)| \]
\[ \geq n - 1 + |N_2(u_1)| + |N_2(u_p)| - |N^-(u_1) \cap N^+(u_p)| \]
\[ \quad \text{since } \sigma_3 \geq n - 1 \]
\[ \geq n - |V(T)\setminus C| \quad \text{by Claim 2.1C} \]

and so \(|C| + |V(T)\setminus C| \geq n\), which is impossible because \( C \) and \( V(T)\setminus C \) are
disjoint and w belongs to neither.

This concludes the proof of Theorem 2.1.

The condition \( \sigma_3(G) \geq n - 1 \) in Theorem 2.1 is best possible. For \( n \geq 4 \), let \( A_n \)
be the graph obtained from \( K_{n-3} \) by choosing a vertex \( v \), then adding new vertices
\( w, x_1, x_2 \) and edges \( vw, wx_1, wx_2 \). These graphs appear in Theorem 1.6, above.
For \( n \geq 5 \), \( A_n \) has \( \sigma_3 = n - 2 \) and no spanning trail of any kind. Moreover, \( K_{1,3} \),
the exceptional graph in Theorem 2.1, is just \( A_4 \). As another family of examples,
for \( m \geq 1 \) the graph \( K_{m,2m+2} \) has \( \sigma_3 = 3m = n - 2 \) but no spanning 2-trail.
Note also that the hypothesis that \( G \) is connected is necessary; it is easy to
construct disconnected graphs with \( \sigma_3 \geq n - 1 \).

We also have the following straightforward corollary, which is sharp, at least
when \( n \equiv 2 \pmod{3} \), as shown by \( K_{m,2m+2} \). As usual, \( \delta(G) \) denotes the minimum
degree of \( G \), and we use the facts that \( \sigma_3(G) \geq 3\delta(G) \) and \( \sigma_3(G) \geq 3\sigma_2(G)/2 \).

**Corollary 2.1.** Let \( G \) be a connected \( n \)-vertex graph. If \( \delta(G) \geq (n - 1)/3 \) or
\( \sigma_2(G) \geq 2(n - 1)/3 \) then \( G \) has a (possibly open) spanning 2-trail unless
\( G \cong K_{1,3} \).

**3. CLOSED 2-TRAILS AND HAMILTON PATHS**

Now we consider the situation where we have a spanning 2-trail, and try to find a
closed spanning 2-trail. The following fact will be used frequently.

**Observation 3.1.** If \( u_1u_2 \cdots u_p \) is a trail and \( u_k = u_l \) with \( k < l \), then \( l \geq k + 3 \).
Lemma 3.1. Suppose Assumption 2.1 holds, and \( u_k = u_l \) with \( k < l \).

(i) Suppose \( k \leq i \leq l \), and \( u_iu_i \) or \( u_iu_i \) is in \( E(G) \setminus E(T) \). If \( i \geq k + 1 \) then \( u_{i-1} \) is used only once by \( T \), and if \( i \leq l - 1 \) then \( u_{i+1} \) is used only once by \( T \).

(ii) None of \( u_iu_{i+1}, u_iu_{i+1}, u_iu_{i-1} \) or \( u_iu_{i-1} \) is in \( E(G) \setminus E(T) \).

(iii) \( u_k = u_i, u_{k-1}, u_{k+1}, u_{l-1}, \) and \( u_{l+1} \) all exist and are distinct vertices.

Proof. (i) Suppose \( k + 1 \leq i \leq l \), and \( u_iu_i \in E(G) \setminus E(T) \). Then there is a spanning 2-trail \( T' = u_{i-1}u_{i-2} \cdots u_{k+1}(u_k = u_i)u_{i+1} \cdots u_{i-1}(u_l = u_k)u_{k-1} \cdots u_i \) with the same number of edges as \( T \). Hence, by Lemma 2.1, \( T' \), and hence \( T \), uses \( u_{i-1} \) only once. The other cases are similar.

(ii) If one of the given edges is in \( E(G) \setminus E(T) \), then by (i) \( u_k = u_l \) would be used only once by \( T \), a contradiction.

(iii) Note that \( k \geq 2 \) and \( l \leq p - 1 \) by Lemma 2.1, so all these vertices exist. All of \( u_{k-1}, u_{k+1}, u_{l-1}, u_{l+1} \) are joined to \( u_k = u_l \) by distinct edges of \( T \), so all five vertices are distinct.

Given a trail \( T = v_1v_2 \cdots v_k \), define \( \rho_{\min}(T) \) to be the position of the first repeated vertex, i.e., the smallest \( i \) such that \( v_i = v_j \) for some \( j \neq i \), or \( \infty \) if \( T \) has no repeated vertices. Similarly, define \( \rho_{\max}(T) \) to be the largest \( i \) such that \( v_i = v_j \) for some \( j \neq i \), or \( -\infty \) if \( T \) has no repeated vertices.

The main result of this section is the following. Note that the proof requires that we take care to distinguish between indices of vertices along \( T \) and the vertices themselves, and between elements of \( E(G) \setminus E(T) \) and of \( E(T) \).

Theorem 3.1. Let \( G \) be a connected \( n \)-vertex graph with \( \sigma_3(G) \geq n \). Then at least one of the following holds.

(i) \( G \) has a hamilton path.

(ii) \( G \) has a closed spanning 2-trail.

Proof. Suppose \( G \) has neither a hamilton path nor a closed spanning 2-trail. Let \( T = u_1u_2 \cdots u_p \) be a spanning 2-trail with (1) fewest edges; (2) subject to (1), smallest \( \rho_{\min}(T) \); and (3) subject to (1) and (2), largest \( \rho_{\max}(T) \). Assumption 2.1 holds for \( T \). Since \( G \) has no hamilton path, \( q = \rho_{\min}(T) \) is finite, and \( u_q = u_t \) for some \( t > q \).

If \( u_1 \) and \( u_p \) are adjacent, then there is a closed spanning 2-trail, so they are nonadjacent. We begin by finding a vertex \( u_{r+1} \) which will form the third vertex of an independent set with \( u_1 \) and \( u_p \).

Claim 3.1A. We may assume (possibly after a reordering of the vertices and edges of \( T \) that does not change \( \rho_{\min}(T) \) or \( \rho_{\max}(T) \) ) that there exists \( r, 2 \leq r \leq p - 4 \), such that

(a) \( u_r = u_s \) for some \( s, r + 3 \leq s \leq p - 1 \).

(b) \( r = q \) or \( q + 1 \), so that each of \( u_1, \ldots, u_{r-2} \) is used only once on \( T \).
(c) Either \( u_{r-1} \) is used only once on \( T \), or \( u_1 u_r \) and \( u_p u_r \) are not in \( E(G) \setminus E(T) \), or both.

(d) \( u_{r+1} \neq u_2 \) and \( u_{r+1} \neq u_{p-1} \).

(e) There is no \( i \geq r + 1 \) such that \( u_{r+1} u_{i+1} \in E(T) \) and at least one of \( u_1 u_i \) or \( u_p u_i \) is in \( E(G) \setminus E(T) \).

**Proof.** Take \( r = q \) (and \( s = t \)). This satisfies (a)–(c) above, since \( u_1, \ldots, u_{q-1} \) are used only once on \( T \). If \( u_{q+1} = u_2 \) then \( u_2 \) is used twice, so \( q = 2 \) giving \( u_{q+1} = u_3 = u_2 \), which is impossible. Thus, \( u_{q+1} \neq u_2 \). If \( u_{q+1} = u_{p-1} \) then, by Lemma 3.1(iii), \( u_{t-1} \neq u_{q+1} = u_{p-1} \). Therefore, by reversing the segment \( u_q u_{q+1} \cdots u_{t-1} u_t \) of \( T \), we may assume that \( u_{q+1} \neq u_{p-1} \), and (d) also holds. We shall show that either (e) holds, or we can make another choice of \( r \).

In our arguments we shall suppose that (e) does not hold. In doing this, we focus on the vertex \( u_{i+1} \) as described in (e). We say “Suppose \( x = u_{i+1} \) as in (e)” to mean “Suppose there is \( i \geq r + 1 \) so that \( x = u_{i+1}, u_{r+1} u_{i+1} \in E(T) \) and at least one of \( u_1 u_i \) or \( u_p u_i \) is in \( E(G) \setminus E(T) \).”

Suppose \( u_q = u_i = u_{i+1} \) as in (e). Since \( i \geq r + 1 = q + 1, i = t - 1 \), which contradicts Lemma 3.1(ii) for \( u_q \ldots u_t \).

Suppose \( u_{q+2} = u_{i+1} \) as in (e). If \( i = q + 1 \) then Lemma 3.1(ii) for \( u_q \cdots u_t \) is contradicted. So \( u_{q+2} = u_{i+1} \) for some \( i \neq q + 1 \). By Observation 3.1, either \( i + 1 \leq q - 1 \) or \( i + 1 \geq q + 5 \). However, the former contradicts \( q = \rho_{\min}(T) \), so \( i + 1 \geq q + 4 \) and in particular \( i + 1 > q + 2 \). Now \( u_1 u_t \) or \( u_p u_t \) contradicts Lemma 3.1(ii) for \( u_{q+2} \cdots u_{i+1} \).

If \( u_{q+1} \) is used only once by \( T \), we are finished. So, assume that \( u_{q+1} \) is used twice, with \( u_{q+1} = u_l \), where \( l \geq q + 4 \) since \( q = \rho_{\min}(T) \).

Suppose \( u_{l+1} = u_{i+1} \) as in (e). If \( i = l \) then \( u_1 u_l = u_1 u_{q+1} \) or \( u_p u_l = u_p u_{q+1} \) contradicts Lemma 3.1(ii) for \( u_q \cdots u_t \). Therefore, \( u_{l+1} = u_{i+1} \) where \( i \neq l \). If \( i > l \) then \( u_1 u_l \) or \( u_p u_l \) contradicts Lemma 3.1(ii) for \( u_{l+1} \cdots u_{i+1} \), so \( i < l \). Since \( q = \rho_{\min}(T), i + 1 \geq q \). By Lemma 3.1(iii), \( u_l = u_{q+1} \), \( u_q \) and \( u_{l+1} = u_{i+1} \) are distinct. Therefore, \( i + 1 \neq q \) or \( q + 1 \), so \( i + 1 \geq q + 2 \), or \( i \geq q + 1 \). Now applying Lemma 3.1(i) to \( u_l \) in \( u_{q+1} \cdots u_t \), we see that \( u_{i+1} \) is used only once by \( T \), a contradiction.

Suppose \( u_{l-1} = u_{i+1} \) as in (e). If \( i = l - 2 \) then we choose a new \( r \) (see below). So, suppose \( i \neq l - 2 \). Then \( u_{l-1} = u_{i+1} \) where \( l - 1 \neq i + 1 \). By an argument similar to the one for \( u_{i+1} = u_{l+1} \), above, we obtain a contradiction to Lemma 3.1(i).

So, we satisfy conditions (a)–(e) with \( r = q \), unless \( u_{q+1} = u_l \) where \( u_1 u_{l-2} \) or \( u_p u_{l-2} \) is in \( E(G) \setminus E(T) \). In that case, reorder \( T \) as \( T' = u_{l+1} u_2 \cdots u_q(u_{q+1} = u_l)u_{i-1} \cdots u_{q+2}(u_{q+1} = u_l)u_{i+1} \cdots u_p = v_1 \cdots v_p \), and take \( r = q + 1 \). Then \( v_r = u_{q+1} = u_l \), and neither \( v_l v_r = u_1 u_{q+1} \) nor \( v_p v_r = u_p u_{q+1} \) are in \( E(G) \setminus E(T) \) by Lemma 3.1(ii) on \( u_q \cdots u_t \). Thus, (a)–(c) are satisfied. Note that by Lemma 3.1(i) for \( u_{l-2} \) on \( u_{q+1} \cdots u_l \) we have that \( T \) uses \( u_{l-1} = v_{r+1} \) only once, so \( T' \) also uses \( v_{r+1} \) only once. For (d), since \( 4 \leq r + 1 \leq p - 2 \) and \( v_{r+1} \) is used only once, \( v_{r+1} \neq v_2 \) or \( v_{p-1} \). For (e), since \( v_{r+1} = u_{l-1} \) is used only once, we
need only show that neither \( v_r = v_l = u_l = u_{q+1} \) nor \( v_{r+2} = u_{i-2} \) can be \( v_{i+1} \) as in (e).

Suppose \( v_r = v_l = v_{i+1} \) as in (e) (applied to \( T' \)). Since \( i \geq r + 1, i = l - 1 \), which contradicts Lemma 3.1(ii) for \( v_r \cdots v_l \).

Suppose \( v_{r+2} = v_{i+1} \) as in (e). If \( i = r + 1 \) then Lemma 3.1(ii) for \( v_r \cdots v_l \) is contradicted. So \( v_{r+2} = v_{i+1} \) for some \( i \neq r + 1 \). Either \( i + 1 \leq r - 1 = q \) or \( i + 1 \geq r + 5 \). Since \( q = \rho_{\min}(T), i + 1 \geq q \). If \( i + 1 = q = r - 1 \), then by Lemma 3.1(i) applied to \( v_{r-1} = v_{r+2} \) on \( v_{r-1} \cdots v_{r+2}, v_r = u_{q+1} \) is used only once on \( T' \), and hence on \( T \), a contradiction. Thus, \( i + 1 \geq r + 5 \) and in particular \( i + 1 > r + 2 \). Now \( v_1v_l \) or \( v_pv_l \) contradicts Lemma 3.1(ii) for \( v_{r+2} \cdots v_{i+1} \).

The two reorderings of \( T \) mentioned above do not change \( \rho_{\min}(T) \) or \( \rho_{\max}(T) \), so this completes the proof of the claim.

Now \( u_{r+1} \) is not adjacent to \( u_1 \) or \( u_p \) by an element of \( E(T) \), by Claim 3.1A(d), or by an element of \( E(G) \setminus E(T) \), by Lemma 3.1(ii) for \( u_r \cdots u_p \). Therefore, \( \{u_1, u_{r+1}, u_p\} \) is an independent set.

For an arbitrary \( v \), define the following sets of integers:

\[
I(v) = \{i \mid i \leq r \text{ and } vu_i \in E(G \setminus E(T))\},
\]

\[
J(v) = \{i \mid i \geq r + 1 \text{ and } vu_i \in E(G \setminus E(T))\}.
\]

Observe that

\[
I(u_1) \subseteq \{3, 4, \ldots, r\}, \quad J(u_1) \subseteq \{r + 1, r + 2, \ldots, p - 1\},
\]

\[
I(u_p) \subseteq \{2, 3, \ldots, r\}, \quad J(u_p) \subseteq \{r + 1, r + 2, \ldots, p - 2\}.
\]

Therefore, for \( v = u_1 \) or \( u_p \), it makes sense to define

\[
A^-(v) = \{u_{i-1} \mid i \in I(v)\},
\]

\[
B^+(v) = \{u_{i+1} \mid i \in J(v)\}.
\]

Then, let

\[
C = N(u_{r+1}) \cup A^-(u_1) \cup A^-(u_p) \cup B^+(u_1) \cup B^+(u_p).
\]

**Claim 3.1B.** \( N(u_{r+1}), A^-(u_1) \cup A^-(u_p), \) and \( B^+(u_1) \cup B^+(u_p) \) are pairwise disjoint.

**Proof.** Suppose first that \( v \) is in both \( A^-(u_1) \cup A^-(u_p) \) and \( B^+(u_1) \cup B^+(u_p) \).

Then \( v = u_{i-1} = u_{j+1} \) where \( i \in I(u_1) \cup I(u_p), j \in J(u_1) \cup J(u_p) \), so that \( i - 1 < r < j + 1 \). Then the edge \( u_1u_i \) or \( u_pu_i \) contradicts Lemma 3.1(ii) on \( u_{i-1} \cdots u_{j+1} \).

Now suppose that \( v \) is in both \( B^+(u_1) \cup B^+(u_p) \) and \( N(u_{r+1}) \). Write \( v = u_{i+1} \) where \( i \in J(u_1) \cup J(u_p) \), so that \( i \geq r + 1 \). By Claim 3.1A(e), \( u_{r+1}u_{i+1} \notin E(T) \).

Since \( u_{r+1}u_s \in E(T), i \neq s - 1 \). We show that in all cases a spanning 2-trail \( T' \)
with fewer edges than $T$ can be found, giving a contradiction. If $r + 1 \leq i \leq s - 2$ and $u_1u_i \in E(G) \setminus E(T)$, let

$$T' = u_pu_{p-1} \cdots u_{s+1}(u_s = u_{r-1})u_{r-1} \cdots u_1u_iu_{i-1} \cdots u_{r+1}u_{i+1}u_{i+2} \cdots u_{s-1}. $$

A similar trail $T'$ can be found if $r + 1 \leq i \leq s - 2$ and $u_pu_i \in E(G) \setminus E(T)$. If $i \geq s$ and $u_1u_i \in E(G) \setminus E(T)$, let

$$T' = u_pu_{p-1} \cdots u_{i+1}u_{i+2} \cdots u_s \cdots u_1u_1u_2 \cdots u_{r-1}. $$

If $i \geq s$ and $u_pu_i \in E(G) \setminus E(T)$, let

$$T' = u_1u_2 \cdots u_{r-1}(u_r = u_s)u_{s+1} \cdots u_1u_pu_{p-1} \cdots u_{i+1}u_{i+2} \cdots u_{s-1}. $$

Now suppose that $v$ is in both $A^-(u_1) \cup A^-(u_p)$ and $N(u_{r+1})$. Write $v = u_{i-1}$, where $i \in I(u_1) \cup I(u_p)$, so that $i \leq r$. If $u_{r+1}u_{i-1} \in E(T)$, then $u_{i-1}$ must be used twice on $T$, so by Claim 3.1A(b), $i = r$. However, since $u_{i-1} = u_{r-1}$ is used twice, by Claim 3.1A(c) $i = r \notin I(u_1) \cup I(u_p)$, which is a contradiction. Thus, $u_{r+1}u_{i-1} \in E(G) \setminus E(T)$, and the result follows by arguments similar to those of the previous paragraph for the case $i \geq s$.

This completes the proof of the claim. □

**Claim 3.1C.** (a) The map $i \mapsto u_{i-1}$ provides bijections $I(u_1) \rightarrow A^-(u_1)$, $I(u_p) \rightarrow A^-(u_p)$, and $I(u_1) \cap I(u_p) \rightarrow A^-(u_1) \cap A^-(u_p)$.

(b) The map $i \mapsto u_{i+1}$ provides bijections $J(u_1) \rightarrow B^+(u_1), J(u_p) \rightarrow B^+(u_p)$, and $J(u_1) \cap J(u_p) \rightarrow B^+(u_1) \cap B^+(u_p)$.

**Proof.** The arguments for $I(u_1) \rightarrow A^-(u_1), I(u_p) \rightarrow A^-(u_p), J(u_1) \rightarrow B^+(u_1)$ and $J(u_p) \rightarrow B^+(u_p)$ are similar; we prove the third. By definition of $B^+(u_1)$, $i \mapsto u_{i+1}$ maps $J(u_1)$ onto $B^+(u_1)$. We prove it is also one-to-one. If not, then there are $i, j \in J(u_1)$ with $i < j$ such that $u_{i+1} = u_{j+1}$. But now $u_1u_j \in E(G) \setminus E(T)$ contradicts Lemma 3.1(ii) for $u_1 \cdots u_{j+1}$.

The arguments for $I(u_1) \cap I(u_p) \rightarrow A^-(u_1) \cap A^-(u_p)$ and $J(u_1) \cap J(u_p) \rightarrow B^+(u_1) \cap B^+(u_p)$ are similar; we prove the latter. The map $i \mapsto u_{i+1}$ maps $J(u_1) \cap J(u_p)$ into $B^+(u_1) \cap B^+(u_p)$, but we must prove that this map is one-to-one and onto. It is one-to-one because it is a restriction of the one-to-one map $J(u_1) \rightarrow B^+(u_1)$. To show it is onto, consider any $v \in B^+(u_1) \cap B^+(u_p)$. Since $v \in B^+(u_1), v = u_{i+1}$ where $i \in J(u_1)$, and since $v \in B^+(u_p), v = u_{j+1}$ where $j \in J(u_p)$. If $i < j$ then $u_pu_j \in E(G) \setminus E(T)$ contradicts Lemma 3.1(ii) for $u_{i+1} \cdots u_{j+1}$, and if $i > j$ we get a similar contradiction. Therefore $i = j \in J(u_1) \cap J(u_p)$ maps to $v = u_{i+1}$. □

We now modify our definitions of $N(v)$ and $N_i(v)$ to get $N^*(v)$ and $N^*_i(v)$, where $N^*(v) = \{w \mid vw \in E(G) \setminus E(T)\}$, and $N^*_i(v)$ contains the vertices of $N^*(v)$ that are used $i$ times on $T, i = 1$ or 2. Thus, $N^*(u_1) = N(u_1) \setminus \{u_2\}$, and $N^*(u_p) = N(u_p) \setminus \{u_{p-1}\}$. 
Claim 3.1D. If \( v = u_1 \) or \( u_p \) then \(|I(v)| + |J(v)| = |N(v)| + |N^*_2(v)| - 1\).

Proof. In both cases \(|N^*(v)| = |N(v)| - 1\), so

\[
|I(v)| + |J(v)| = \{|i \mid u/ui \in E(G) \setminus E(T)\}| \\
= |N^*_1(v)| + 2|N^*_2(v)| = |N^*(v)| + |N^*_2(v)| \\
= |N(v)| + |N^*_2(v)| - 1. 
\]

Claim 3.1E. (a) If \( i \in (I(u_1) \cap I(u_p)) \cup (J(u_1) \cap J(u_p)) \) then \( u_i \in N^*_2(u_1) \cap N^*_2(u_p) \).

(b) \(|I(u_1) \cap I(u_p)| + |J(u_1) \cap J(u_p)| = 2|N^*_2(u_1) \cap N^*_2(u_p)|\).

Proof. (a) By definition, \( u_i \in N^*(u_1) \cap N^*(u_p) \). If \( u_i \) is not used twice, then \( u_1u_2 \cdots u_pu_1u_iu_1 \) is a closed spanning 2-trail, which is a contradiction.

(b) Each element of \( N^*_2(u_1) \cap N^*_2(u_p) \) has the form \( u_i \) for exactly two numbers \( i \in (I(u_1) \cap I(u_p)) \cup (J(u_1) \cap J(u_p)) \). By (a), these numbers \( i \) include all elements of \( (I(u_1) \cap I(u_p)) \cup (J(u_1) \cap J(u_p)) \).

Claim 3.1F. \( u_{r+1} \notin C \).

Proof. Clearly \( u_{r+1} \notin N(u_{r+1}) \). If \( u_{r+1} \in A^-(u_1) \cup A^-(u_p) \), then \( u_{r+1} = u_{i-1} \) where \( i \leq r \). By Claim 3.1A(b), we must have \( i - 1 = r - 1 \), which contradicts Observation 3.1. If \( u_{r+1} \in B^+(u_1) \cup B^+(u_p) \), then \( u_{r+1} = u_{i+1} \) where \( i \geq r + 1 \). Then the edge \( u_1u_i \) or \( u_pu_i \) in \( E(G) \setminus E(T) \) contradicts Lemma 3.1(ii) for \( u_{r+1} \cdots u_{i+1} \).

Claim 3.1G. We have

\[
|V(G)\setminus \{u_{r+1}\}|C| + |N^*_2(u_p)\setminus N^*_2(u_1)| + |N^*_2(u_1)\setminus N^*_2(u_p)| \leq 1.
\]

Proof. We have

\[
|C| = |N(u_{r+1})| + |A^-(u_1) \cup A^-(u_p)| + |B^+(u_1) \cup B^+(u_p)| \\
\text{by Claim 3.1B} \\
= |N(u_{r+1})| + |A^-(u_1)| + |A^-(u_p)| - |A^-(u_1) \cap A^-(u_p)| \\
+ |B^+(u_1)| + |B^+(u_p)| - |B^+(u_1) \cap B^+(u_p)| \\
= |N(u_{r+1})| + |I(u_1)| + |I(u_p)| - |I(u_1) \cap I(u_p)| \\
+ |J(u_1)| + |J(u_p)| - |J(u_1) \cap J(u_p)| \quad \text{by Claim 3.1C} \\
= |N(u_{r+1})| + |N(u_1)| + |N(u_p)| + |N^*_2(u_1)| + |N^*_2(u_p)| - 2 \\
- |I(u_1) \cap I(u_p)| - |J(u_1) \cap J(u_p)| \quad \text{by Claim 3.1D} \\
\geq n + |N^*_2(u_1)| + |N^*_2(u_p)| + |I(u_1) \cap I(u_p)| - |J(u_1) \cap J(u_p)| - 2 \\
\text{since } \sigma_3 \geq n \\
= n + |N^*_2(u_1)| + |N^*_2(u_p)| - 2|N^*_2(u_1) \cap N^*_2(u_p)| - 2 \\
\text{by Claim 3.1E} \\
= n + |N^*_2(u_p)\setminus N^*_2(u_1)| + |N^*_2(u_1)\setminus N^*_2(u_p)| - 2
\]
and the result follows since \(|V(G)\{u_{r+1}\}\setminus C| = n - |C| - 1\), by Claim 3.1F.

Now we complete the proof of Theorem 3.1. Recall that \(q = \rho_{\min}(T) \geq 2\), and let \(a = \rho_{\max}(T) \leq p - 1\), so that \(u_{a+1}, u_{a+2}, \ldots, u_p\) are used only once. Let \(\alpha_1 = \(|\{u_1, u_2, \ldots, u_{q-1}\}\setminus C|, \alpha_2 = \(|\{u_{a+1}, u_{a+2}, \ldots, u_p\}\setminus C|, \beta_1 = |N_2^+(u_p)\setminus \{u_2\}|, \) and \(\beta_2 = |N_2^+(u_1)\setminus \{u_{p-1}\}|. \) Because \(u_2 \notin N^+(u_1)\) and \(u_{p-1} \notin N^+(u_p),\) Claim 3.1G implies that \(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leq 1\).

Suppose \(\alpha_1 + \beta_1 = 0\). Then \(\alpha_1 = 0\), so \(u_1, u_2, \ldots, u_{q-1} \in C\). Since \(u_1 \in C\) but \(u_2 \notin N(u_{r+1}), A^-(u_1), B^+(u_1)\) or \(B^+(u_p),\) we get \(u_1 \in A^-(u_p),\) and hence \(u_2 \in N^+(u_p).\) Since \(\beta_1 = 0, u_2\) must be used only once on \(T.\) Hence \(q \geq 3.\) We shall prove that \(u_1, u_2, \ldots, u_{q-2} \in A^-(u_p)\). If \(q = 3\) we are finished. Suppose that \(2 \leq j \leq q - 2,\) and we have proved that \(u_1, \ldots, u_{j-1} \in A^{-}(u_p)\). Since \(u_{j-1} \in A^{-}(u_p), u_j \in N^+(u_p)\). We know \(u_j\) is in \(C,\) but \(u_j \notin B^+(u_1) \cap B^+(u_p).\) If \(u_j \in A^{-}(u_1)\) then \(u_{j+1} \in N^+(u_1)\) and hence

\[ T' = u_1 u_2 \cdots u_j u_p \cdots u_{j+1} u_1 \]

is a closed 2-trail, a contradiction. If \(u_j\) is \(N(u_{r+1})\) then

\[ T' = u_1 u_2 \cdots u_{r-1}(u_r = u_5)u_{s+1} \cdots u_{p-1} u_j u_{p+1} u_{r+2} \cdots u_{s-2} \]

has \(\rho_{\min}(T') = j < q,\) contradicting our choice of \(T.\) Therefore \(u_j \in A^-(u_p).\)

Repeating this argument, we get \(u_{q-2} \in A^-(u_p)\) which means that \(u_{q-1} \in N^+(u_p).\) But then, since \(u_q = u_t\) with \(t > q,\)

\[ T' = u_1 u_2 \cdots u_{q-1} u_p \cdots u_{q-1} u_{q+1} \]

has fewer edges than \(T,\) a contradiction.

Suppose now that \(\alpha_1 + \beta_1 = 1.\) Then \(\alpha_2 + \beta_2 = 0.\) By reasoning symmetric to the above we can show that \(u_p \in B^+(u_1),\) so that \(u_{p-1} \in N^+(u_1),\) and \(u_{p-1}\) is used only once, so that \(a = \rho_{\max}(T) \leq p - 2.\) If \(a = p - 2,\) we can use an argument symmetric to the one above, so suppose that \(a \leq p - 3.\) We know that \(u_{p-1} \in C.\)

By an argument symmetric to the one above, \(u_{p-1} \notin B^+(u_1).\) Suppose that \(u_{p-1} \in N(u_{r+1}).\) Consider the trail

\[ T' = u_{r-1} u_{r-2} \cdots u_{p-1} u_{r+1} u_{r+2} \cdots u_s \cdots u_{p-1} u_p. \]

Considering \((T')^{-1},\) the reverse of \(T',\) we have \(\rho_{\min}((T')^{-1}) = 2\) and therefore \(q = \rho_{\min}(T) = 2\) by choice of \(T.\) Also, by Lemma 2.1, \(u_{r-1}\) is used only once on \(T'\) and hence only once on \(T,\) so by Claim 3.1A(b), \(r = q = 2.\) Now we see that \(\rho_{\min}(T') = 2 = \rho_{\min}(T)\) and \(\rho_{\max}(T') = p - 1 > \rho_{\max}(T).\) This contradicts our choice of \(T.\) Therefore \(u_{p-1} \in B^+(u_1),\) so that \(u_{p-2} \in N^+(u_1).\) But then
the trail

\[ T' = u_2 u_3 \cdots u_{p-2} u_1 u_{p-1} u_p \]

has \( \rho_{\min}(T') = q - 1 < \rho_{\min}(T) \), a contradiction.

This completes the proof of Theorem 3.1.

The condition \( \sigma_3(G) \geq n \) in Theorem 3.1 is best possible. For all \( n_1, n_2, n_3 \geq 1 \) the graph \( K_1 + (K_{n_1} \cup K_{n_2} \cup K_{n_3}) \) (‘+’ denotes join) has \( \sigma_3 = n_1 + n_2 + n_3 = n - 1 \) but neither a hamilton path nor a closed spanning 2-trail. These graphs also appear as sharpness examples in reference [10]. Also, for every \( m \geq 1 \) the graph \( K_{m,2m+1} \) has \( \sigma_3 = 3m = n - 1 \) but neither a hamilton path nor a closed spanning 2-trail. Note also that the hypothesis that \( G \) is connected is necessary; it is easy to construct disconnected graphs with \( \sigma_3 \geq n \).

We also have the following straightforward corollary, which is sharp, at least when \( n \equiv 1 \pmod{3} \), as shown by \( K_{m,2m+1} \).

**Corollary 3.1.** Let \( G \) be a connected \( n \)-vertex graph. If \( \delta(G) \geq n/3 \) or \( \sigma_2(G) \geq 2n/3 \) then \( G \) has either a hamilton path or a closed spanning 2-trail.

It is also interesting to note the following, which follows from Theorem 3.1 because a graph with a cutedge cannot have a closed spanning trail.

**Corollary 3.2.** If \( G \) is an \( n \)-vertex graph with a cutedge and \( \sigma_3(G) \geq n \) then \( G \) has a hamilton path.

## 4. CLOSED 2-TRAILS IN 2-EDGE-CONNECTED GRAPHS

In this section we prove our third main result, as follows.

**Theorem 4.1.** Let \( G \) be a 2-edge-connected \( n \)-vertex graph with \( \sigma_3(G) \geq n \). Then either

(i) \( G \) is isomorphic to \( K_{2,3} \), or the 6-vertex graph \( K_{2,3}^* \) obtained by subdividing an edge of \( K_{2,3} \), or

(ii) \( G \) has a closed spanning 2-trail.

**Proof.** The result holds for \( n = 1 \) (whether or not one considers \( K_1 \) to be 2-edge-connected), so suppose that \( n \geq 2 \). Assume that \( G \) does not have a closed spanning 2-trail. By Theorem 3.1, \( G \) has a hamilton path \( T = u_1 u_2 \cdots u_p \), where now \( p = n \), and \( u_1 u_p \notin E(G) \). Since \( G \) is 2-edge-connected, \( \deg(u_1), \deg(u_p) \geq 2 \), so that \( N^*(u_1), N^*(u_p) \neq \emptyset \), and \( n \geq 4 \).

Define

\[ \eta(T) = \min\{ j - i \mid u_i \in N^*(u_1), u_j \in N^*(u_p) \} , \]

which may be positive, zero or negative.

**Claim 4.1A.** There exists a hamilton path \( T \) with \( \eta(T) \leq 0 \).
Proof. Suppose not. Let $T$ be a Hamilton path with $\eta(T)$ as small as possible. Then there exist $q, t$ with $t - q = \eta(T) > 0$ such that $u_q \in N^+(u_1)$, $q \geq 3$, $N(u_1) \subseteq \{u_2, u_3, \ldots, u_q\}$, $u_t \in N^+(u_p)$, $t \leq p - 2$, and $N(u_p) \subseteq \{u_t, u_{t+1}, \ldots, u_{p-1}\}$.

Suppose first that there exists some $r \leq q - 1$ such that $u_r u_t \in E(G)$ for some $s \geq q + 1$. Then $r \geq 2$. Choose $r$ as large as possible. If $r = q - 1$ then

$$T' = u_{s-1} u_{s-2} \cdots u_{q-1} u_1 u_2 \cdots u_{q-1} u_s u_{s+1} \cdots u_p$$

has $\eta(T') \leq t - s < \eta(T)$, a contradiction. So $r \leq q - 2$.

Define

$$A^-(u_1) = \{u_{i-1} \mid i \leq r, u_i u_1 \in E(G)\},$$

$$B^+(u_1) = \{u_{i+1} \mid i \geq r + 1, u_1 u_i \in E(G)\}.$$ 

Clearly $|A^-(u_1)| + |B^+(u_1)| = |N(u_1)|$.

Now we claim that $A^-(u_1), B^+(u_1), N(u_{r+1})$ and $N^+(u_p)$ are pairwise disjoint. Since $A^-(u_1) \subseteq \{u_1, \ldots, u_{r-1}\}$, $B^+(u_1) \subseteq \{u_{r+2}, \ldots, u_{q+1}\}$, and $N^+(u_p) \subseteq \{u_{r+1}, \ldots, u_p\}$, these three sets are disjoint; furthermore, $u_{r+1}$ belongs to none of them. Therefore we need only consider intersections of $N(u_{r+1})$ with the other three sets. If there is $u_{i-1} \in A^-(u_1) \cap N(u_{r+1})$, then

$$T' = u_{s-1} u_{s-2} \cdots u_{r+1} u_{i-1} u_i \cdots u_{q-1} u_s u_{s+1} \cdots u_p$$

has $\eta(T') \leq t - s < \eta(T)$, a contradiction (this works even if $i = 2$). If there is $u_{i+1} \in B^+(u_1) \cap N(u_{r+1})$, then because $r + 1 \leq q - 1$ and we chose $r$ as large as possible, $i \leq q - 1$, and

$$T' = u_{s-1} u_{s-2} \cdots u_{i+1} u_{r+1} u_{r+2} \cdots u_i u_{i+1} u_{i+2} \cdots u_r u_s u_{s+1} \cdots u_p$$

has $\eta(T') \leq t - s < \eta(T)$, a contradiction. If there is $u_{i+1} \in N^+(u_p) \cap N(u_{r+1})$ then because $r + 1 \leq q - 1$, we contradict our choice of $r$ as large as possible.

Since $u_1 \in A^-(u_1)$ and $u_p \in N^+(u_p)$, the above shows us that $\{u_1, u_{r+1}, u_p\}$ is independent. We therefore have

$$n \leq |N(u_1)| + |N(u_p)| + |N(u_{r+1})|$$

$$= |A^-(u_1)| + |B^+(u_1)| + |N^+(u_p)| + |N(u_{r+1})|$$

which is impossible, since these four sets are disjoint and $u_{r+1}$ belongs to none of them.

Therefore, there is no number $r$ as described above, and $u_q$ is a cutvertex. Making the same argument with $T$ reversed, which does not change $\eta(T)$, we find that $u_t$ is also a cutvertex. If $t = q + 1$ then $u_q u_t$ is a cutedge, which is a contradiction. Thus, $t \geq q + 2$. Now $\{u_1, u_{q+1}, u_p\}$ is independent, and it is easy to see that $\deg(u_1) + \deg(u_{q+1}) + \deg(u_p) \leq n - 1$, again a contradiction.

Hence, there must exist $T$ with $\eta(T) < 0$. 

\[ \square \]
Now for each hamilton path $T$ define
\[
\theta(T) = \min\{i - j \mid u_j \in N^+(u_{p}), u_i \in N^-(u_{1}), j \leq i\},
\]
which is finite when $\eta(T) \leq 0$, and $\infty$ (the minimum of the empty set) when $\eta(T) > 0$. Let $T$ be a hamilton path with $\theta(T)$ as small as possible. By Claim 4.1A, $\theta(T)$ is finite, so there exist $r, s$ with $s - r = \theta(T), u_r \in N^+(u_{p})$ and $u_s \in N^-(u_{1})$. If $\theta(T) = 0$, we have a closed spanning 2-trail $u_1u_2 \cdots u_pur_{r=s}u_1$, and if $\theta(T) = 1$, then we have a hamilton cycle $u_1u_2 \cdots ur_{r+1=s}u_1$.

So we may assume that $\theta(T) \geq 2$, i.e., $s \geq r + 2$. None of $ur_{r+1}, ur_{r+2}, \ldots, us_{s-1}$ are adjacent to $u_1$ or $u_p$, by definition of $\theta(T)$. For $v = u_1$ or $u_p$, define
\[
A^-(v) = \{u_{i-1} \mid i \leq r, vu_i \in E(G)\},
\]
\[
B^+(v) = \{u_{i+1} \mid i \geq s, vu_i \in E(G)\}.
\]
so that $|A^-(v)| + |B^+(v)| = |N(v)|$. Also, let
\[
C = N(u_{r+1}) \cup A^-(u_1) \cup A^-(u_p) \cup B^+(u_1) \cup B^+(u_p).
\]

**Claim 4.1B.** $A^-(u_1), A^-(u_p), B^+(u_1), B^+(u_p)$, and $N(u_{r+1})$ are pairwise disjoint, except that possibly $u_1 \in A^-(u_1) \cap A^-(u_p)$, and possibly $u_p \in B^+(u_1) \cap B^+(u_p)$.

**Proof.** Since $A^-(u_1), A^-(u_p) \subseteq \{u_1, \ldots, ur_{r-1}\}$ and $B^+(u_1), B^+(u_p) \subseteq \{us_{s+1}, \ldots, up\}$, the former two are disjoint from the latter two. If there is a vertex other than $u_1$ in $A^-(u_1) \cap A^-(u_p)$, or other than $u_p$ in $B^+(u_1) \cap B^+(u_p)$, then there is $uj \in N^+(u_1) \cap N^+(u_p)$ and
\[
T' = u_1u_2 \cdots upuju_1
\]
is a closed 2-trail, a contradiction.

Now consider the intersections of $N(u_{r+1})$ with the other sets. If there is $ui_{i-1} \in A^-(u_1) \cap N(u_{r+1})$ then
\[
T' = u_1 \cdots ui_{i-1}ur_{r+1}ur_{r+2} \cdots upur_{r-1} \cdots uiu_1
\]
is a hamilton cycle, a contradiction (this works even when $i = 2$). If there is $ui_{i-1} \in A^-(u_p) \cap N(u_{r+1})$ then
\[
T' = u_1u_2 \cdots ui_{i-1}ur_{r+1}ur_{r+2} \cdots us \cdots upuiu_{i+1} \cdots ur
\]
has $\theta(T') \leq s - r - 1 < \theta(T)$ because of the edges $u_1us, ur_{r+1}$, which is a contradiction. If there is $ui_{i+1} \in B^+(u_1) \cap N(u_{r+1})$, then
\[
T' = u_1u_2 \cdots ur_{r}up_{r-1} \cdots ui_{i+1}ur_{r+1}ur_{r+2} \cdots uiu_1
\]
is a hamilton cycle, a contradiction. If there is $ui_{i+1} \in B^+(u_p) \cap N(u_{r+1})$ then
\[
T' = u_1u_2 \cdots ui_{i}up_{i-1} \cdots ui_{i+1}
\]
has $\theta(T') \leq s - r - 1 < \theta(T)$ because of the edges $u_1 u_s, u_{i+1} u_{r+1}$, which is a contradiction (this works even when $i = p - 1$). Thus, $N(u_{r+1})$ is disjoint from the other four sets.

This ends the proof of the claim. \hfill \blacksquare

**Claim 4.1C.** $|V(G)\setminus C| \leq |A^-(u_1) \cap A^-(u_p)| + |B^+(u_1) \cap B^+(u_p)|$.

**Proof.** By Claim 4.1B, $N(u_{r+1}), A^-(u_1) \cup A^-(u_p)$ and $B^+(u_1) \cup B^+(u_p)$ are pairwise disjoint, so

$$|C| = |N(u_{r+1})| + |A^-(u_1) \cup A^-(u_p)| + |B^+(u_1) \cup B^+(u_p)|$$

$$= |N(u_{r+1})| + |A^-(u_1)| + |A^-(u_p)| - |A^-(u_1) \cap A^-(u_p)| + |B^+(u_1)| + |B^+(u_p)| - |B^+(u_1) \cap B^+(u_p)|$$

$$= |N(u_{r+1})| + |N(u_1)| + |N(u_p)| - |A^-(u_1) \cap A^-(u_p)|$$

$$\geq n - |A^-(u_1) \cap A^-(u_p)| - |B^+(u_1) \cap B^+(u_p)|$$

and the result follows. \hfill \blacksquare

Let $\gamma = |V(G)\setminus C|, \alpha = |A^-(u_1) \cap A^-(u_p)|$ and $\beta = |B^+(u_1) \cap B^+(u_p)|$. Claim 4.1C says that $\gamma \leq \alpha + \beta$. Since $u_{r+1} \in V(G)\setminus C$, we know that $\gamma \geq 1$.

By Claim 4.1B, $\alpha = 1$ if $u_2 u_p \in E(G) \setminus E(T)$, and 0 otherwise, while $\beta = 1$ if $u_1 u_{p-1} \in E(G) \setminus E(T)$, and 0 otherwise.

**Claim 4.1D.** $\theta(T) = 2$, i.e., $s = r + 2$.

**Proof.** Suppose that $\theta(T) \geq 3$, i.e., $s \geq r + 3$. If $u_s \in N(u_{r+1})$, then

$$T' = u_1 u_2 \cdots u_s u_p u_{p-1} \cdots u_{r+1} u_s u_1$$

is a closed 2-trail, a contradiction. Thus, $u_s \notin N(u_{r+1})$ and hence $u_s \in V(G)\setminus C$ and $\gamma \geq 2$. Therefore $\alpha = \beta = 1$, so that $u_2 u_p, u_1 u_{p-1} \in E(G) \setminus E(T)$. But now

$$T' = u_3 u_4 \cdots u_p u_2 u_1$$

is a hamilton path with $\theta(T') \leq 2 < \theta(T)$ because of the edges $u_3 u_2, u_1 u_{p-1}$, which is a contradiction. \hfill \blacksquare

**Claim 4.1E.** Suppose $1 \leq k \leq r - 1, u_k \in A^-(u_p)$, and $u_{k+1}, \ldots, u_{r-1} \in C$. Then for $k \leq j \leq r - 1$, we have $u_j \in A^-(u_p)$, and for $k + 1 \leq j \leq r - 1$ we have $u_j \notin N(u_{r+1}) \cup A^-(u_1)$.

**Proof.** We show that $u_j \in A^-(u_p)$ for $k \leq j \leq r - 1$ by induction on $j$, proving the rest along the way. We are given $u_k \in A^-(u_p)$. Suppose now that
\(k + 1 \leq j \leq r - 1\). By induction, \(u_{j-1} \in A^- (u_p)\), so \(u_j \in N^*(u_p)\). If \(u_j \in N(u_{r+1})\), then

\[
T' = u_1 u_2 \cdots u_{r+1} u_j u_p u_{p-1} \cdots u_{r+2} = u_1
\]
is a closed spanning 2-trail, a contradiction. If \(u_j \in A^- (u_1)\) then

\[
T' = u_1 u_2 \cdots u_j u_p u_{p-1} \cdots u_{j+1} u_1
\]
is a hamilton cycle, a contradiction. Therefore \(u_j \notin N(u_{r+1}) \cup A^- (u_1)\), but since \(u_j \in C\), we must have \(u_j \in A^- (u_p)\).

**Claim 4.1F.** If \(\alpha = 1\) and \(u_2, \ldots, u_{r-1} \in C\), then

- (a) \(u_1, u_2, \ldots, u_{r-1} \in A^- (u_p)\), and consequently \(u_2, u_3, \ldots, u_r \in N^*(u_p)\);  
- (b) \(N^*(u_1) \subseteq \{u_{r+2}, u_{r+4}, u_{r+5}, \ldots, u_{p-1}\}\);  
- (c) \(N^*(u_{r+1}) \subseteq \{u_{r+4}, \ldots, u_{p-1}\}\).

**Proof.** Since \(\alpha = 1\), we apply Claim 4.1E with \(k = 1\).

(a) This follows immediately.

(b) Clearly \(u_1, u_2 \notin N^*(u_1)\). By Claim 4.1E, \(u_2, u_3, \ldots, u_{r-1} \notin A^- (u_1)\), so \(u_3, \ldots, u_r \notin N^*(u_1)\). By definition of \(\theta(T)\), \(u_{r+1} \notin N^*(u_1)\). If \(u_{r+3} \in N^*(u_1)\) then

\[
T' = u_2 u_3 \cdots u_{r+2} u_1 u_{r+3} u_{r+4} \cdots u_p u_2
\]
is a hamilton cycle, a contradiction. We know that \(u_p \notin N^*(u_1)\).

(c) We know that \(u_1, u_p \notin N^*(u_{r+1})\). By Claim 4.1E, \(u_2, u_3, \ldots, u_{r-1} \notin N^*(u_{r+1})\). Clearly \(u_r, u_{r+1}, u_{r+2} \notin N^*(u_{r+1})\). If \(u_{r+3} \in N^*(u_{r+1})\), then

\[
T' = u_1 u_2 \cdots u_p u_{p-1} \cdots u_{r+3} u_{r+1} u_{r+2} u_1
\]
is a hamilton cycle, a contradiction.

Now we complete the proof of Theorem 4.1. Since \(\theta(T) = 2\), i.e., \(s = r + 2\), we have \(r + 1 = s - 1\). This means that symmetric arguments may be applied from the two ends of our path \(T\). We examine two cases.

**Case I.** \(\alpha = 0\) and \(\beta = 1\), or \(\alpha = 1\) and \(\beta = 0\). By symmetry we may suppose that \(\alpha = 1\) and \(\beta = 0\). Then \(\gamma = 1\), so \(u_{r+1}\) is the only vertex that does not belong to \(C\). By Claim 4.1F(b) and the fact that \(\beta = 0\), \(N^*(u_1) \subseteq \{u_{r+2}, u_{r+3}, \ldots, u_{p-2}\}\), and since \(N^*(u_1) \neq \emptyset\), \(r \leq p - 4\).

Suppose that \(r \leq p - 5\).

First, we show that \(u_{r+4}, \ldots, u_p \notin B^+(u_1)\). We know \(u_p \notin B^+(u_1)\) since \(\beta = 0\). If there is \(u_k \in B^+(u_1)\) with \(r + 4 \leq k \leq p - 1\), then by Claim 4.1E (using
symmetry) we get $u_{r+3}, \ldots, u_k \in B^+(u_1)$. In particular, since $k \geq r + 4$, $u_{k-1} \in B^+(u_1)$, and

$$T' = u_2u_3 \cdots u_{k-2}u_1u_{k-1}u_k \cdots uPu_2$$

is a hamilton cycle, a contradiction. From Claim 4.1F(b), we now know that $N^*(u_1) = \{u_{r+2}\}$.

Second, we show that if $r \leq p - 6$, then $u_{r+4}, \ldots, u_{p-2} \notin N(u_{r+1})$. Suppose there is $k, r + 4 \leq k \leq p - 2$, with $u_k \in N(u_{r+1})$. If $u_{k+1} \in B^+(u_p)$ then

$$T' = u_1u_2 \cdots u_{r+1}u_ku_pu_{p-1} \cdots u_{r+1}u_1$$

is a closed 2-trail, a contradiction. From above, $u_{k+1} \notin B^+(u_1)$. Since $u_{k+1} \in C$, we must have $u_{k+1} \in N(u_{r+1})$. Now

$$T' = u_1u_2 \cdots u_rup_{p-1} \cdots u_{k+1}u_{r+1}u_ku_{k-1} \cdots u_{r+2}u_1$$

is a hamilton cycle, a contradiction. From Claim 4.1F(c), we now know that $N^*(u_{r+1}) \subseteq \{u_{p-1}\}$.

Third, we show that $u_{p-1} \notin N(u_{r+1})$. Suppose $u_{p-1} \in N(u_{r+1})$. If $r \geq 3$, then, by Claim 4.1F(a), $u_2, u_3 \in N^*(u_p)$, and

$$T' = u_1u_2u_3u_4 \cdots u_{r+1}u_{p-1}u_{p-2} \cdots u_{r+2}u_1$$

is a hamilton cycle, a contradiction. Therefore, $r = 2$. If $r \leq p - 6$, then from above $u_{r+4}$ must be in $B^+(u_p)$ since it is not in $B^+(u_1) \cup N(u_{r+1})$, and

$$T' = u_1u_2 \cdots u_rup_{r+3}u_{r+4} \cdots u_{p-1}u_{r+1}u_{r+2}u_1$$

is a hamilton cycle, a contradiction. Therefore, $r = p - 5$. Thus, $p = n = 7$. Besides $T, G$ contains the edges $u_1u_4, u_2u_7$ and $u_3u_6$. There are no further edges incident with $u_1$ or $u_{r+1} = u_3$. If this is all of $G$, then $\{u_1, u_5, u_7\}$ is an independent set with degree sum $6 < n = 7$, a contradiction. Therefore, there is an additional edge incident with at least one of $u_5$ or $u_7$, either to increase the degree sum or destroy the independence of this set. The only possible extra edges are $u_5u_2, u_5u_7$ and $u_7u_4$. The edge $u_5u_7$ gives a hamilton cycle $u_1u_2u_3u_6u_7u_5u_4u_1$, and the edge $u_7u_4$ gives the closed 2-trail $u_1u_2 \cdots u_7u_4u_1$, all of which provide contradictions.

So, now we know that $u_{r+4}, \ldots, u_{p-1} \in B^+(u_p)$ and $N^*(u_{r+1}) = \emptyset$. Thus, $\{u_1, u_{r+1}, u_{p-1}\}$ is an independent set. Since $\deg(u_1) = \deg(u_{r+1}) = 2$, $\deg(u_{p-1}) \geq n - 4$, so $u_{p-1}$ is nonadjacent to at most three vertices, and hence adjacent to at least one of $u_r$ or $u_{r+2}$. If $u_{p-1}u_r \in E(G)$, then

$$T' = u_1u_{r+1}u_{r+2}u_1u_2 \cdots u_rup_{r+3}u_{r+4} \cdots u_{p-1}u_r$$
is a closed 2-trail, a contradiction. If \( u_{p-1}u_{r+2} \in E(G) \), then

\[
T' = u_1u_2 \cdots u_{r+2} \cdots u_{p-2}up_{p-1}u_{r+2}u_1
\]

is a closed 2-trail, a contradiction. Thus, we do not have \( r \leq p - 5 \).

Therefore, \( r = p - 4 \). Then \( N(u_1) = \{u_2, u_{r+2}\} \). By Claim 4.1F(c), \( N^*(u_{r+1}) = \emptyset \) and so \( N(u_{r+1}) = \{u_r, u_{r+2}\} \). Since \( \beta = 0, u_1 \notin N^*(u_{p-1}) \). Suppose \( u_j \in N^*(u_{p-1}) \) where \( 2 \leq j \leq r \). By Claim 4.1(a), \( u_j \in N^*(u_p) \), so

\[
T' = u_1u_2 \cdots u_ju_{p-1}up_ju_{j+1} \cdots u_{r+2}p_{p-2}u_1
\]

is a closed 2-trail, a contradiction. We know \( u_{r+1} \notin N^*(u_{p-1}) \). Therefore, \( N(u_{p-1}) = \{u_{r+2} = p-2, u_p\} \). Now \( \{u_1, u_{r+1}, u_{p-1}\} \) is an independent set with degree sum 6, so \( p = n \leq 6 \). Since \( r = p - 4 \geq 2 \), we have \( p = 6 \) and \( r = 2 \). Besides \( T, G \) contains the edges \( u_1u_4 \) and \( u_2u_6 \). If this is all of \( G \), then \( G \cong K_{2,3} \).

Otherwise, since \( \{u_1, u_{r+1}, u_p\} = \{u_1, u_3, u_6\} \) and \( \{u_1, u_{r+1}, u_{p-1}\} = \{u_1, u_3, u_5\} \) are independent, \( G \) contains the edge \( u_2u_4 \), and has a closed spanning 2-trail \( u_1u_2u_3u_4u_5u_6u_2u_4u_1 \), a contradiction.

Case II. \( \alpha = \beta = 1 \). Since there is at most one vertex of \( V(G) \setminus C \) other than \( u_{r+1} \), at least one of \( D_1 = \{u_1, u_2, \ldots, u_{r-1}\} \) or \( D_2 = \{u_{r+1}, u_{r+2}, \ldots, u_p\} \) is a subset of \( C \). By symmetry we may suppose that \( D_1 \subseteq C \).

If \( r \geq 3 \), then by Claim 4.1(a) we have \( u_2, u_3 \in N^*(u_p) \), and

\[
T' = u_1u_2u_pu_3u_4 \cdots u_{p-1}u_1
\]

is a hamilton cycle, a contradiction.

Therefore, \( r = 2 \). If \( u_3u_{p-1} \notin E(G) \), then

\[
T' = u_3u_2u_1u_4u_5 \cdots u_p
\]

is a hamilton path with \( 2 = \theta(T) \leq \theta(T') \leq 2 \) that falls under case I; so we are finished. If \( u_3u_{p-1} \in E(G) \setminus E(T) \) then

\[
T' = u_1u_{p-1}u_3u_4 \cdots u_{p-1}u_pu_2u_1
\]

is a closed 2-trail, a contradiction. Therefore, \( u_3u_{p-1} \in E(T) \) and \( p = 5 \). Besides \( T, G \) contains the edges \( u_1u_4 \) and \( u_2u_5 \). If this is all of \( G \), then \( G \cong K_{2,3} \).

Otherwise, since \( \{u_1, u_3, u_5\} \) is independent, \( G \) contains the edge \( u_2u_4 \) and has a closed spanning 2-trail \( u_1u_2u_3u_4u_5u_2u_4u_1 \), a contradiction.

This concludes the proof of Theorem 4.1. 

The condition \( \sigma_3(G) \geq n \) in Theorem 4.1 is best possible. Suppose \( n \geq 7 \). Take \( K_n \), choose (possibly identical) vertices \( v_1, v_2 \), then add new vertices
$w_1, w_2, x_1, x_2$ and edges $v_1w_1, v_2w_2, w_1x_1, w_1x_2, w_2x_1, w_2x_2$. We get two graphs, depending on whether $v_1$ is equal to $v_2$ or not. Both are 2-edge-connected (the one with $v_1 \neq v_2$ is actually 2-connected), have $\sigma_3 = n - 1$, and do not have a closed spanning 2-trail. Also, for every $m \geq 2$ the graph $K_{m, 2m + 1}$ is 2-edge-connected (actually, $m$-connected), has $\sigma_3 = 3m = n - 1$, and has no closed spanning 2-trail. Note also that the hypothesis that $G$ is 2-edge-connected is clearly necessary.

We also have the following straightforward corollary, which is sharp, at least when $n \equiv 1 (\mod 3)$, as shown by $K_{m, 2m + 1}$. This shows that Theorem 4.1 strengthens Theorem 1.4.

**Corollary 4.1.** Let $G$ be a 2-edge-connected $n$-vertex graph. If $\delta(G) \geq n/3$ or $\sigma_2(G) \geq 2n/3$ then $G$ has a closed spanning 2-trail unless $G \cong K_{2, 3}$ or $K_{2, 3}^*$.

Results for graphs embedded on surfaces can also be obtained from minimum degree conditions. An argument similar to that of Duke [8], using Euler’s formula to show that $\delta \geq n/3$, gives the following. We omit the details. Similar results can be obtained from Corollaries 2.1 and 3.1.

**Corollary 4.2.** Suppose $G$ is a 2-edge-connected graph embeddable on a surface (compact, without boundary, orientable or nonorientable) of Euler characteristic $\chi < 0$, such that $\delta(G) \geq 3 + \sqrt{9 - 2\chi}$. Then $G$ has a closed spanning 2-trail.

Perhaps Theorem 4.1 can be extended to spanning $k$-trails for $k \geq 3$. The natural conjecture would be that $\sigma_{k+1} \geq n$, together with sufficient edge-connectivity, guarantees the existence of a closed spanning $k$-trail. Another interesting question is whether a toughness condition, together with sufficient edge-connectivity, guarantees the existence of a closed spanning $k$-trail. Similar results are known for closed spanning $k$-walks for $k \geq 2$ [9,12], and a result for $k$-trails might shed some light on Chvátal’s well known conjecture [7] that sufficiently tough graphs have a hamilton cycle.

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