An analysis of an optimal control problem for mosquito populations

Cícero Alfredo da Silva Filho\textsuperscript{a,}\textsuperscript{*}, José Luiz Boldrini\textsuperscript{b}

\textsuperscript{a} Universidade Estadual de Santa Cruz, Ilhéus, BA, Brazil
\textsuperscript{b} Universidade Estadual de Campinas, Campinas, SP, Brazil

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\textbf{ABSTRACT}

We analyze a system of nonlinear partial differential equations modeling the dynamics of certain mosquito populations by taking in consideration the iterations among the immature (aquatic) subpopulation, the adult winged subpopulation and the environment resources; the immature subpopulation is assumed to be age-structured and the model also considers the action of control mechanisms on these subpopulations.

After a first analysis on the existence and uniqueness of solutions for the model, we use the obtained results to prove the existence of an optimal solution of a given optimal control problem. The corresponding first order optimality conditions are also rigorously obtained by approximating our original optimal control problem by pure optimization problems obtained by using penalization arguments.

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\textbf{1. Introduction}

In this article we analyze an optimal control problem associated to the following system of partial differential equations related to the one studied in Calsina e Elidrissi [1]:

\begin{equation}
\begin{cases}
u_t(a,t) + u_a(a,t) + m_1(r(t))u(a,t) + \mu_1(c(a,t))u(a,t) = 0, \\
v'(t) + m_2(r(t))v(t) + \mu_2(L_1(c))v(t) = u(l,t), \\
r'(t) - [g(r(t)) - h(L_2(u,v))]r(t) = 0, \\
u(0,t) = b(t)v(t), \\
u(a,0) = u_0(a), \\
v(0) = v_0, \\
r(0) = r_0.
\end{cases}
\end{equation}

This system models the dynamics of a mosquito populations by taking in consideration the interaction among the immature (aquatic) form, the adult (winged) form of the mosquitos and the amount of available resources for survival.

\textsuperscript{*} Corresponding author.

E-mail addresses: cicero@uesc.br (C.A. da Silva Filho), josephbold@gmail.com (J.L. Boldrini).

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The first equation is of Gurtin–MacCamy type and governs the age-structured dynamics of the immature (aquatic) mosquito population, \( u = u(a,t) \), where \( a \) represents age, and \( t \), time; here, \( 0 \leq a \leq l \), where \( l > 0 \) is given and denotes the maturation age, that is, the age when an aquatic individual becomes adult; \( T > 0 \) is given and denotes the maximum time of interest; we denote \( Q = (0,l) \times (0,T) \). We remark that, for simplicity of the model, eggs, pupae and larval forms were lumped together in a unique aquatic population.

The second equation governs the dynamics of the adult (winged) mosquito population, \( v = v(t) \), which in the present model, as in Calsina e Elidrissi [1], is considered non-age-structured.

The third equation governs the variation of the amount of available resources denoted by \( r(t) \).

The fourth equation in (1) is the standard renewal condition that requires that new immature individuals enter in the system due to reproduce mechanism of the adults; in this equation, \( b(\cdot) \in L^\infty(0,T) \) is a positive function related to the adult fertility rate.

Moreover, as in Calsina e Elidrissi [1], in the first equation of (1), \( m_1(r(t)) \) is the natural mortality rate of aquatic individuals, while \( m_2(r(t)) \) in the second equation is the corresponding natural mortality rate of the adults; both such mortalities may depend on the available amount of resources. In the third equation, \( g(\cdot) \) is a Verhurzt type function associated to the possibility of recovery of the available resources, while the degradation rate of the resources by the populations is given by \( h(\cdot) \), which is a known nonnegative function; such degradation is mediated by a linear integral operator \( L_2 \) to be described in the next section.

However, differently from the system in Calsina e Elidrissi [1], here we consider the action of an external control variable \( c = c(a,t), (a,t) \in Q \), associated for instance with the use of chemical agents, that act by increasing the mortality rates of the populations; in the case of immature individuals, such action may depend on their respective maturity level. Thus, in the first equation of (1), \( \mu_1(c(a,t)) \) is an additional mortality rate of the immature form caused by the external control \( c \), while in the second equation \( \mu_2(L_1(c))(t) \) is the respective additional mortality rate for the adult form; the control action in this case is mediated by a linear integral operator \( L_1 \) to be detailed described in the next section. In other words, (1) is a controlled version of the model considered by [1].

We also observe that we set the controls in such way that, by choosing suitably the given data, we may consider several situations. In particular we also have the extreme situations: by taking \( \mu_1(\cdot) \neq 0 \) and \( \mu_2(\cdot) \equiv 0 \), we are left with a problem where just the aquatic immature subpopulation is being controlled; on the contrary, by taking \( \mu_1(\cdot) \equiv 0, \mu_2(\cdot) \neq 0 \) and calling \( \tilde{c} = L_1(c)(t) \) as the actual control, we are left with a problem where just the adult winged subpopulation is being controlled; in other situations both subpopulations are affected by the control.

To finish the description of the system data, we say that as expected from their meanings, the initial conditions are as follows: \( u_0 \) is a nonnegative function in \( L^\infty(0,l) \); \( v_0 \) and \( r_0 \) are nonnegative numbers.

In the first part of this work, we will be concerned with the question about existence, uniqueness and estimates of such solutions. In this aspect, we recall that Calsina e Elidrissi [1], by using semigroup theory, presented a result on global existence and uniqueness of solutions for the system without the control terms and somewhat different requirements (for instance, they assume that the death rates of young and adults are decreasing functions of the amount of available resources); see also related results in Calsina and Elidrissi [2] and in Calsina and Saldaña [3]. Moreover, these authors were mainly interested in the asymptotic behavior of the solutions.

On the other hand, here we are interested in obtaining existence and uniqueness results suitable for using in optimal control problems with (1) as the state equations. With this purpose, we present a result of this type for system (1), under rather general conditions and in a sense to be precisely described in the next section. Such result is presented not only because the system analyzed in this work is a controlled one, and our conditions are somewhat different than the ones of Calsina and Elidrissi [1], but mainly because it is prepared to be used in optimal control problems with (1) as the state equations, and thus a very important aspect to be analyzed are the explicit dependence of certain estimates for such solutions on the parameters.
of the problem and the controls acting on the system since this is crucial in showing the existence of optimal solutions. We also remark that for obtaining our results, instead of using techniques similar to the ones in [1], we use fixed point arguments.

To exemplify the significance of the present result on existence of solutions and related estimates, in the second part of this work we consider the question of existence of optimal controls for the following optimal control problem associated to (1): we want to show the existence of a control \( c^* \in U \) such that

\[
F(c^*) = \min \{ F(c) : c \in U \}. \tag{2}
\]

Here, the functional \( F(c) \) to be minimized is

\[
F(c) = \rho_0 \int_0^T \int_0^l G(a,u(a,t)) \, dt \, dt + \rho_1 \int_0^T \int_0^l |c(a,t)|^{p_1} \, dt \, dt
+ \bar{\rho}_1 \int_0^T \int_0^l |c_i(a,t)|^{\tilde{p}_1} \, dt \, dt + \rho_2 \int_0^T \int_0^l |c_a(a,t)|^{p_1} \, dt \, dt
+ \rho_2 \int_0^T |v(t)|^{p_2} \, dt + \rho_3 \int_0^T |r(t)|^{p_3} \, dt, \tag{3}
\]

where \((u,v,r)\) is the solution of (1) associated to the \( c \) being considered; \( G : (0,l) \times \mathbb{R} \to \mathbb{R} \) is a given lower bounded function such that \( G(a,y) \) is measurable in \( a \) for each fixed \( y \) and continuous and convex in \( y \) for each fixed \( a \); the weighting constants \( \rho_1, \bar{\rho}_1, \tilde{p}_1 > 0 \) and \( \rho_2, \rho_3 \geq 0 \) and the exponents \( p_1, \tilde{p}_1, \bar{p}_1, p_2, p_3 \geq 1 \) are given.

As usual, expression (3) to be minimize, through suitable choices of the weighting constants tries to balance the minimization of several relevant variables in the problem; the first term in (3) is associated to the minimization of the aquatic mosquito population; the second, third and fourth, to the cost of control (amount of application of chemical agents and changes in such amounts); the fifth, to the minimization of the aquatic mosquito population; the sixth, to the available resources. We remark expression (3) is rather general and includes situations that one does not want to minimize the available resources; for this is enough to take \( \rho_3 = 0 \), which is a case allowed in our results.

To define the set of admissible controls, \( U \), consider the Banach space \( \mathcal{B} = \{ c \in L^{p_1}(Q) : c_i \in L^{\tilde{p}_1}(Q), c_a \in L^{\bar{p}_1}(Q) \} \), with norm given by \( \| c \|^2_{\mathcal{B}} = \| c \|^2_{L^{p_1}(Q)} + \| c_i \|^2_{L^{\tilde{p}_1}(Q)} + \| c_a \|^2_{L^{\bar{p}_1}(Q)} \), and define \( U \) as:

\[
U \text{ is a closed convex set in } \mathcal{B}. \tag{4}
\]

The actual values eventually attributed to the nonnegative weighting constants \( \rho_0, \rho_1, \bar{\rho}_1, \tilde{p}_1, \rho_2, \rho_3 \) are related to the relative importance that one desires to assign to each one of the terms in (3) in the minimization. The constant \( \rho_0 \) gives the relative weight considered to the minimization of the immature mosquito population; \( \rho_2 \) gives the relative weight of the winged adult mosquito population; \( \rho_3 \) gives the relative weight considered for the reduction of the available resources for the mosquito population; \( \rho_1 \) penalizes the functional by assigning a relative weight of the control cost (like financial or ambient hazards); \( \bar{\rho}_1 \) and \( \tilde{p}_1 \) also penalize sudden changes in the control policy.

There are a huge scientific literature on optimal control problems; for general mathematical techniques, we just recall the interesting book by Barbu [4]. As for optimal control problems for age-structured evolution equations, there are also many published works (most of them with the dynamics governed by scalar equations); here, we just mention a few representative ones. Closely related to the aspects studied in the present article are the book by Anita [5] (and the references there in), and the articles by Anita, Iannelli, Kim and Park [6] and Barbu and Iannelli [7]. Interesting works considering other situations are the following: Ainseba, Anita and Langlais [8], Alexandrian et al. [9], Belyakov and Veliov [10,11], Brokate [12], Cai, Modnak and Wang [13], Chan, Guyatt, Bundy and Medley [14], Demasse, Tewa, Bowong and Emvudu [15],
Feichtinger, Prskawetz and Veliov [16], Feichtinger, Tragler and Velio [17], Fister and Lenhart [18], Jacob, Omrane [19], Müller [20], Picart, Ainseba and Milner [21], Sharomi and Malik [22], Simon, Skritek and Veliov [23]; an interesting and fairly complete survey up to 2010 can be found in Greenhalgh [24].

Concerning the mathematical techniques to be employed here, to prove the existence of optimal controls for the present problem, in Section 5, we will use the direct method of calculus of variations, that is, we will employ minimization sequences; however, the estimates given in our existence and uniqueness theorem for (1) will not be enough to allow us to pass certain limits in the nonlinear terms, and it will be necessary to obtain further ones that allow us to complete the task. After that, in Section 6, we will rigorously obtain the first order optimality conditions for our optimal control problem by approximating by pure optimization problems obtained by using penalization arguments.

Finally, we remark that the existence result and estimates presented here (Theorem 2) are also essentially used in an article presently under preparation, da Silva and Boldrini [25], where it is analyzed a non convex optimal control problem with a cost functional similar to the one in Barbu and Iannelli [7] and dynamics given by system (1) instead of just a scalar Gurtin–MacCamy type equation as in [7]. In this non convex situation instead of using just arbitrary minimizing sequences as we did in the present convex case, we have to use a special minimizing sequence obtained by using the Ekeland’s variational principle (see for instance Ekeland [26]).

2. Preliminaries and main results

We use standard notations for the functional spaces; in particular, we will use Banach spaces of type $L^p(0,T;B) = \{ f : (0,T) \rightarrow B : \|f\|_{L^p(0,T;B)} < +\infty \}$, where $B$ is a suitable Banach space and the norm is given by $\|f\|_{L^p(0,T;B)} = \| \|f(t)\|_B \|_{L^p(0,T)}$. We remark that $L^p(Q) = L^p((0,T);L^p(0,l))$. Results for Sobolev spaces $W^s_p(0,l)$, as well more general ones, including functional analysis, can be found for instance in Adams [27], Evans [28], Folland [29] and Brezis [30].

We denote $C^+[0,T] = \{ f \in C[0,T] : f(\cdot) \geq 0 \}$ and $L^\infty_+(0,T,L^1(0,l)) = \{ f \in L^\infty(0,T,L^1(0,l)) : f(\cdot) \geq 0 \}$.

Technical hypotheses and notations:

Moreover, throughout the proof of existence of solutions of (1), we suppose that:

(H1) Related to the first equation in (1), we assume that: The mortality rates of the immature form $m_1, \mu_1 : \mathbb{R} \rightarrow [0,\infty)$ are Lipschitz functions with Lipschitz constants respectively denoted $k_{m_1}, k_{\mu_1}$. The adult fertility rate in the boundary condition is a nonnegative function $b \in L^\infty(0,T)$; the maximum fertility rate is

$$b_m = \|b\|_{L^\infty(0,T)} > 0.$$  \hfill (5)

The initial condition for the immature mosquito population is a nonnegative function $u_0 \in L^\infty(0,l)$.

(H2) Related to the second equation in (1), we assume that:

The mortality rates of the adult form $m_2, \mu_2 : \mathbb{R} \rightarrow [0,\infty)$ are Lipschitz functions with Lipschitz constants respectively denoted $k_{m_2}, k_{\mu_2}$. The operator $L_1 : U \subset L^{p_1}(Q) \rightarrow L^{p_1}(0,T)$, with $p_1 \geq 1$, is defined for each $c \in U$ by

$$L_1(c)(t) = \int_0^1 c(a,t)H_0(a,t)da,$$  \hfill (6)

where $H_0 \in L^\infty(Q)$ and $H_0(\cdot) \geq 0$.

The initial condition for the adult mosquito population is a nonnegative constant $v_0$. 
(H3) Related to the third equation in (1), we assume that:

The growth rate \( g : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function of Verhulstian type, that is, there exists a constant \( r_c > 0 \) such that \( g(z) \geq 0 \) for \( z \leq r_c \) and \( g(z) < 0 \) for \( z > r_c \) (thus \( g(r_c) = 0 \)). We also assume that \( k_g \geq 0 \) is a constant such that \( |g(z_2) - g(z_1)| \leq k_g|z_2 - z_1| \) for any \( z_1, z_2 \in [0, r_c] \).

We denote the natural maximum growth rate by

\[
g_m = \max\{g(z) : 0 \leq z \leq r_c\}. \tag{7}
\]

The function \( h : \mathbb{R} \to \mathbb{R} \), associated to the consumption of available resources, is a Lipschitz function with Lipschitz constant \( k_h \) and such that \( h(z) \geq 0 \) for \( z \geq 0 \).

The operator \( L_2 : L^\infty([0, T], L^1(0, l)) \times C[0, T] \to L^\infty(0, T) \) is defined by

\[
L_2(u, v)(t) = \int_0^t [u(a, t)H_1(a, t) + v(t)H_2(a, t)]da, \tag{8}
\]

where \( H_1, H_2 \in L^\infty(Q) \) and \( H_1(\cdot), H_2(\cdot) \geq 0 \).

The initial condition is a constant \( r_0 \) such that \( 0 \leq r_0 \leq r_c \).

For the existence of optimal controls, we will additionally assume:

(H4) Concerning the assumptions on system (1): The function \( h \) defined in (H3) is \( h(z) = c_0z \), with \( c_0 > 0 \), for \( z \in \mathbb{R} \).

(H5) Concerning the set of admissible controls \( U \):

\( U \) is a convex set which is closed in \( L^{p_1}(Q) \), where \( p_1 \geq 1 \) is the same as in (H2).

(H6) Concerning the functional \( F \) to be minimized and defined in (3):

\( G : (0, l) \times \mathbb{R} \to \mathbb{R} \) is a lower bounded function such that \( G(a, y) \) is measurable in \( a \) for each fixed \( y \) and continuous and convex in \( y \) for each fixed \( a \). Moreover, there exist \( q > 1 \) and nonnegative constants \( A \) and \( B \) such that

\[
G(a, y) \leq A + B|y|^q. \tag{9}
\]

For the consideration of the optimal control problem, the exponent \( p_1 \) is taken to be the same as the one in (H2); however, we require here that the exponents satisfy \( 1 < p_1, \bar{p}_1, \tilde{p}_1, p_2, p_3 < +\infty \), and moreover

\[
1 + \frac{1}{p_1} - \frac{1}{q_1} > 0 \quad \text{where } q_1 = \min\{p_1, \bar{p}_1\}. \tag{10}
\]

The weighting constants satisfy \( \rho_0, \rho_1, \tilde{\rho}_1 > 0 \) and \( \rho_2, \rho_3 \geq 0 \).

We will show that system (1) has a unique solution in the following sense:

**Definition 1.** Given \( c \in L^{p_1}(Q) \), a triple \((u, v, r) \in L^\infty(0, T, L^1(0, l)) \times C[0, T] \times C[0, T] \), with \( v \) and \( r \) absolutely continuous in \([0, T]\) is called a solution if

\[
\begin{align*}
&\begin{cases}
  u(a, t) = \left\{ \begin{array}{ll}
    u_0(a - t)e^{-\int_0^t[m_1(r(t-\sigma)) + \mu_1(c(a-\sigma, t-\sigma))]d\sigma} & \text{for } a \geq t, \\
    b(t)v(t - a)e^{-\int_0^a[m_1(r(t-\sigma)) + \mu_1(c(a-\sigma, t-\sigma))]d\sigma} & \text{for } a < t,
  \end{array} \right. \\
  v'(t) + m_2(r(t))v(t) + \mu_2(L_1(c))v(t) = u(l, t), \\
  \text{in Caratheodory sense on the time interval } [0, T], \\
  r'(t) - [g(r(t)) - h(L_2(u, v))]r(t) = 0, \\
  \text{in Caratheodory sense on the time interval } [0, T], \\
  v(0) = v_0, \\
  r(0) = r_0.
\end{cases}
\end{align*} \tag{11}
\]
The first equation is understood in the sense of $L^s(Q)$, with $s = \min\{q_1, \tilde{p}_1\}$; its precise meaning is the following: for $Q_1 = \{(a,t) \in Q: a \geq t\}$ and $Q_2 = \{(a,t) \in Q: a < t\}$, then
\[
u|_{Q_1} = u_0(a-t)e^{-\int_0^t[m_1(r(t-\sigma)) + \mu_1(c(a-\sigma,t-\sigma))]d\sigma},
\]
\[
u|_{Q_2} = b(t)v(t-a)e^{-\int_0^t[m_1(r(t-\sigma)) + \mu_1(c(a-\sigma,t-\sigma))]d\sigma}.
\]

**Remark 1.** Since the expression for $\nu$ is obtained by the method of characteristics, $\nu$ is in fact absolutely continuous along the characteristics of the first equation in (1), that is along the straight lines of form $t = a + c$ with $c$ constants.

Concerning the existence of solutions of system (1) in the previous sense, we have the following result:

**Theorem 2.** Suppose that hypotheses (H1)–(H3) hold true; then system (1) has a unique solution. Moreover, the following estimates hold: for any $a \in [0,l]$ and any $t \in [0,T]$,
\[
0 \leq u(a,t) \leq (1 + b_mT^eT)|u_0|_{L^\infty(0,l)} + b_m(1 + b_mT^T)v_0,
\]
\[
0 \leq v(t) \leq (T^{e_b^mT})|u_0|_{L^\infty(0,l)} + (1 + T^{b_m^mT})v_0,
\]
\[
0 \leq r(t) \leq r_c.
\]

**Remark 2.** These results imply in particular that the first equation in (1) holds in the sense of distribution and that $u_t(a,t) + u_q(a,t) = -m_1(r(t))u(a,t) - \mu_1(c(a,t))u(a,t) \in L^\infty((0,l) \times (0,T)) \subset L^2((0,l) \times (0,T))$. Similarly, $v_t \in L^\infty(0,T) \subset L^2(0,T)$ and $r_t \in L^\infty(0,T) \subset L^2(0,T)$. These remarks will be used in Section 6.

As for the previously described optimal control problem, we have:

**Theorem 3.** Suppose that hypotheses (H1)–(H6) hold true, and also that $F$ is defined in (3); $U$ is defined in (4), and the dynamics governed by (1); then problem (2) has an optimal solution $c^* \in U$.

With a bit more of regularity of the parameters of the problem, and for simplicity in the case that $p_1 = \tilde{p}_1 = \tilde{p}_1 = p_2 = p_3 = 2$, the following first order optimality conditions involving Lagrange multipliers hold:

**Theorem 4.** Suppose that the conditions of Theorem 3 hold true with $p_1 = \tilde{p}_1 = \tilde{p}_1 = p_2 = p_3 = q = 2$, and that $c^*$ is an admissible optimal control with corresponding optimal states $u^*$, $v^*$ and $r^*$; that is, they satisfy the dynamical equations (1) with $c = c^*$. Additionally suppose that $m_1(\cdot), m_2(\cdot), g(\cdot), \mu_1(\cdot), \mu_2(\cdot)$ and $G(\cdot, \cdot)$ are $C^1$-functions and $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are bounded. Then, there are adjoint (co-state) variables $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)})$ satisfying the following differential equations
\[
-[(\lambda^{(1)})_t + (\lambda^{(1)})_a] + \lambda^{(1)}(m_1(r^{(c^*)}) + \mu_1(c^*)) - \lambda^{(3)}h'(L_2(u^*, v^*))r^*H_1
\]
\[
+ \rho_0G(\cdot, u^*)G_u(\cdot, u^*) = 0,
\]
\[
-(\lambda^{(2)})' + \lambda^{(2)}(m_2(r^*)) + \mu_2(L_1(c^*)) - \lambda^{(3)}h'(L_2(u^*, v^*))r^*(\int_0^lH_2(a,t)da)
\]
\[
+ b\lambda^{(4)} + 2\rho_2v^* = 0,
\]
\[
-(\lambda^{(3)})' + (\int_0^l\lambda^{(1)}u^*da)m_1'(r^*) + \lambda^{(2)}m_2'(r^*)v^*
\]
\[
+ \lambda^{(3)}[-g'(r^*)r^* - g(r^*) + h(L_2(u^*, v^*))] + 2\rho_3r^*,
\]
\[
\lambda^{(4)}(\cdot) - \lambda^{(1)}(0, \cdot) = 0,
\]
together with the following final and boundary conditions
\begin{align}
\lambda^{(1)}(\cdot, T) &= 0, \quad \lambda^{(2)}(T) = 0, \quad \lambda^{(3)}(T) = 0, \\
\lambda^{(1)}(l, \cdot) - \lambda^{(2)}(\cdot) &= 0,
\end{align}
(14)
and also the following differential inequality
\begin{align}
2\rho_1 \int_Q c^*(\hat{c} - c^*)dadt + 2\hat{\rho}_1 \int_Q c^*_1(\hat{c}_1 - c^*_1) \, dt \\
+ 2\hat{\rho}_1 \int_Q c^*_a(\hat{c}_a - c^*_a) \, dt + \int_Q \lambda^{(1)} \mu_1^*(c^*)u^*(\hat{c} - c^*) \, dt \\
+ \int_0^T \lambda^{(2)} \mu_2^*(L_1(c^*))v^*(L_1(\hat{c} - c^*)) \, dt \geq 0.
\end{align}
(15)
Eqs. (13)–(15) together with (1) constitute the first order optimality condition for problem (2).

**Remark 3.** First order optimality conditions are used to formulate numerical algorithms to find optimal controls and their corresponding optimal states. Thus, the previous result can be useful in specific situations to numerically calculate the optimal amount of chemical agents to be applied to control mosquito populations.

3. Auxiliary problems

Before proving our main results, let us introduce the following auxiliary result:

**Proposition 5.** Let \( \tau \in C^+[0, T] \) be fixed and consider the system
\begin{align}
\begin{cases}
u_2(a, t) + u_0(a, t) + m_1(\tau(t))u(a, t) + \mu_1(c(a, t))u(a, t) = 0, \\
v^*(t) + m_2(\tau(t))v(t) + \mu_2(L_1(c(a, t)))v(t) = u(l, t), \\
u(0, t) = b(t)v(t), \\
u(a, 0) = u_0(a), \\
v(0) = v_0.
\end{cases}
\end{align}
(16)

Under hypotheses (H1) and (H2), system (16) has a unique solution \((u, v) \in X = L_+^{\infty}(0, T, L^1(0, l)) \times C^+[0, T].

**Proof.** We take \( 0 < T_1 \leq T \) and consider the complete metric space \( X_1 = L_+^{\infty}(0, T_1, L^1(0, l)) \times C^+[0, T_1] \) with the standard metric
\[
d_{X_1}((u_2, v_2), (u_1, v_1)) = \|u_2 - u_1\|_{L^\infty(0, T_1, L^1(0, l))} + \|v_2 - v_1\|_{C[0, T_1]}.
\]

Consider also the operator defined by
\[
\tau_1 : X_1 \rightarrow X_1, \\
(u, v) \mapsto \tau_1(u, v) = (u, v)
\]
where \((u, v)\) is the solution of:
\begin{align}
\begin{cases}
u_2(a, t) + u_0(a, t) + m_1(\tau(t))u(a, t) + \mu_1(c(a, t))u(a, t) = 0, \\
v^*(t) + m_2(\tau(t))v(t) + \mu_2(L_1(c(a, t)))v(t) = u(l, t), \\
u(0, t) = b(t)v(t), \\
u(a, 0) = u_0(a), \\
v(0) = v_0.
\end{cases}
\end{align}
(17)

We will show that the operator \( \tau_1 \) has a unique fixed point for \( T_1 \) small enough.
For this, we recall that $c \in L^{p_{1}}(Q)$ is given and observe that $\mu_1(c) \in L^{p_{1}}(Q)$; moreover, the first equation in (17) is decoupled from the second since its boundary condition in the third equation in (17) is now a known function. Thus it can be easily solved by using the methods of characteristics given a unique solution $u \in L^\infty_+(0,T,L^1(0,l))$, which is absolutely continuous along the characteristics; this procedure gives:

$$u(a,t) = \begin{cases} 
  u_0(a-t)e^{-\int_{t}^{a} m_1(\tau(t-\sigma)) + \mu_1(c(a-\sigma,t-\sigma))d\sigma} & \text{if } a \geq t \\
  b(t)\tau(t-a)e^{-\int_{a}^{t} m_1(\tau(t-\sigma)) + \mu_1(c(a-\sigma,t-\sigma))d\sigma} & \text{if } a < t.
\end{cases} \quad (18)$$

By substituting this in the second equation of (17) and observing the properties stated in (H2), we then can easily solve it to get a unique solution $v \in C^+[0,T]$ given by

$$v(t) = v_0e^{-\int_{0}^{t} m_2(\tau(\xi)) + \mu_2(L_1(c))d\xi} + \int_{0}^{t} u(l,\sigma)e^{-\int_{\sigma}^{l} m_1(\tau(t-\sigma)) + \mu_1(c(a-\sigma,t-\sigma))d\sigma} d\sigma \quad (19)$$

From the last two expressions, we then conclude that the operator $\tau_1$ is well defined.

To show that $\tau$ has a unique fixed point, we observe that given $\tau(\overline{u}_1, \overline{v}_1) = (u_1, v_1)$ and $\tau(\overline{u}_2, \overline{v}_2) = (u_2, v_2)$, we have:

$$|u_2(a,t) - u_1(a,t)| \leq b|\overline{v}_2(t-a) - \overline{v}_1(t-a)| \leq b||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} \quad \text{when } a < t,$$

$$|u_2(a,t) - u_1(a,t)| = 0 \quad \text{when } a \geq t.$$

For $0 < t < l$, we obtain

$$\int_{0}^{l} |u_2(a,t) - u_1(a,t)|da = \int_{0}^{l} |u_2(a,t) - u_1(a,t)|da + \int_{t}^{l} |u_2(a,t) - u_1(a,t)|da$$

$$\leq b_m \int_{0}^{l} ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} da \leq b_m T_1 ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} ,$$

where $b_m$ is the maximum adult fertility rate defined in (5).

For $0 < l < t$, we get

$$\int_{0}^{l} |u_2(a,t) - u_1(a,t)|da \leq b_m T_1 ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} .$$

In any case, we then can write

$$||u_2 - u_1||_{L^\infty(0,T_1,L^1(0,l))} \leq b_m T_1 ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} . \quad (20)$$

On the other hand,

$$|v_2(t) - v_1(t)| \leq \int_{0}^{t} |u_2(l,\sigma) - u_1(l,\sigma)|d\sigma \leq b_m T_1 ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} ,$$

which implies

$$||v_2 - v_1||_{C[0,T_1]} \leq b_m T_1 ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} . \quad (21)$$

From (20) and (21), we obtain

$$||u_2 - u_1||_{L^\infty(0,T,L^1(0,l))} + ||v_2 - v_1||_{C[0,T_1]} \leq 2b_m T_1 ||\overline{v}_2 - \overline{v}_1||_{C[0,T_1]} ,$$

and thus, by taking $T_1 = \min\{1/(4b_m), T\}$, we in particular get

$$d_{X_1}(\tau_1(\overline{u}_2, \overline{v}_2), \tau_1(\overline{u}_1, \overline{v}_1)) \leq (1/2) \, d_{X_1}(\overline{u}_2, \overline{v}_2), (\overline{u}_1, \overline{v}_1).$$
Therefore, $\tau_1$ has a unique fixed point in $X_1$ which is a solution of system (16) on the time interval $[0, T_1]$.

Next, we repeat all the previous arguments, but now with the initial conditions $(u(\cdot, T_1), v(T_1))$ instead of $(u_0(\cdot), v_0)$, together with the functional space $X_2 = L^\infty(T_1, T_2, L_1(0, l)) \times C^+[T_1, T_2]$ instead of $X_1$ and an operator $\tau_2$ defined similarly as $\tau_1$. Exactly with the same computations then apply in this case, we conclude that by taking $T_2 = \min\{T_1 + 1/(4b_m), T\}$, $\tau_2$ is again a strict contraction with contraction constant again equal to 1/2; this gives once more a unique fixed point which extends the previously obtained solution of (16) to the interval $[T_1, T_2]$. By proceeding successively in this way on the intervals $[T_2, T_3], [T_3, T_4], \ldots$, all of them with size $1/(4b_m)$, except the last one, we extend the solution. Thus, in a finite number of steps we finally obtain a unique solution (16) to the interval $[0, T]$. □

Next, we consider a second auxiliary problem.

**Proposition 6.** Let be given $(u, v) \in X = L^\infty(0, T, L_1(0, l)) \times C^+[0, T]$; then, under hypotheses (H3), problem

$$
\begin{aligned}
\begin{cases}
  r' - [g(r) - h(L_2(u, v))]r(t) = 0, \\
  r(0) = r_0,
\end{cases}
\end{aligned}
$$

has a unique solution $r \in C^+[0, T]$. In fact, $r(\cdot)$ is absolutely continuous and $0 \leq r(t) \leq r_c$ for all $t \in [0, T]$.

**Proof.** Since $h(L_2(u, v)) \in L^\infty(0, T)$ due to the hypotheses in (H1) and (H3), Theorem 5.3, p. 30, in Hale [31] for ordinary differential equations with Carathéodory conditions guarantees the existence and uniqueness of a maximal local absolutely continuous solution of (22) on a maximal interval contained in $[0, T]$.

To prove that this solution is global, that is, it is defined on $[0, T]$, it is enough to prove that $0 \leq r(t) \leq r_c$ for all $t$ in the maximal interval.

For this, we firstly multiply the equation in (22) by $r_-(t)$, the negative part of $r(t)$ (recall that $r(t) = r_+(t) - r_-(t)$; after some computations, one gets:

$$
\frac{1}{2} \frac{d}{dt}(r_-(t))^2 = [g(r(t)) - h(L_2(u(t), v(t)))(r_-(t))^2,
$$

which implies $(r_-(t))^2 = (r_-(0))^2 \exp \left(2 \int_0^t [g(r(s)) - h(L_2(u(s), v(s)))]ds\right)$; since $r(0) = r_0 \geq 0$, we have $r_-(0) = 0$, and so $r_-(t) = 0$, that is, $r(t) \geq 0$ for all $t$.

Next, we rewrite the equation in (22) as:

$$
(r - r_c)' - [g(r) - h(L_2(u, v))](r - r_c) - g(r)r_c + h(L_2(u, v))r_c = 0.
$$

By multiplying this last equation by $(r(t) - r_c)_+$, the positive part of $r(t) - r_c$, after some computations, we get:

$$
\frac{1}{2} \frac{d}{dt}((r(t) - r_c)_+)^2 = [g(r(t)) - h(L_2(u(t), v(t)))(r(t) - r_c)_+)^2
$$

$$
+ r_c g(r(t))(r(t) - r_c)_+ + r_c h(L_2(u(t), v(t))(r(t) - r_c)_+ 
$$

$$
\leq [g(r(t)) - h(L_2(u(t), v(t)))(r(t) - r_c)_+)^2
$$

since $r_c g(r(t))(r(t) - r_c)_+ \leq 0$ and $-r_c h(L_2(u(t), v(t))(r(t) - r_c)_+ \leq 0$ by the hypotheses in (H3).

Thus, by Gronwall’s inequality, we obtain

$$
((r(t) - r_c)_+)^2 \leq ((r(t) - r_c)_+(0))^2 \exp \left(2 \int_0^t [g(r(s)) - h(L_2(u(s), v(s)))]ds\right),
$$

and since $(r(t) - r_c)_+(0) = (r_0 - r_c)_+ = 0$, we conclude that $(r(t) - r_c)_+(t) = 0$ for all $t$, that is, $r(t) \leq r_c$ for all $t$, concluding the proof. □
Now we can state the following:

**Proposition 7.** Under hypotheses (H1), (H2) and (H3), given \( r \in C[0,T] \) such that \( 0 \leq r(t) \leq r_c \), the system

\[
\begin{aligned}
\begin{cases}
  u_t(a,t) + u_a(a,t) + m_1(\tau(t))u(a,t) + \mu_1(c(a,t))a(a,t) = 0, \\
  v'(t) + m_2(\tau(t))v(t) + \mu_2(L_1(c(a,t)))v(t) = u(l,t), \\
  r'(t) - (g(r) - h(L_2(u,v)))r(t) = 0,
\end{cases}
\end{aligned}
\]

\( (23) \)

has a unique solution \( (u,v,r) \in L^\infty(0,T, L^1(0,l)) \times C^+[0,T] \times C^+[0,T] \).

Moreover, this solution satisfies \( \forall (a,t) \in Q \) the following estimates:

\[
\begin{aligned}
0 & \leq u(a,t) \leq \delta_1, \\
0 & \leq v(t) \leq \delta_2, \\
0 & \leq r(t) \leq r_c
\end{aligned}
\]

\( (24) \quad (25) \quad (26) \)

where

\[
\delta_1 = (1 + b_m T e^T)\|u_0\|_{L^\infty(0,l)} + b_m (1 + b_m T e^T)v_0,
\]

\( (27) \)

\[
\delta_2 = (Te^{b_m T})\|u_0\|_{L^\infty(0,l)} + (1 + Tb_m e^{b_m T})v_0.
\]

\( (28) \)

**Proof.** The existence and uniqueness of solutions, as well the fact their components are nonnegative functions, immediately follows from Propositions 5 and 6; (26) also follows from Proposition 6.

Let us now prove the other estimates. Again by the method of characteristics, we have

\[
u(a,t) = \begin{cases}
  u_0(a-t)e^{-\int_0^t[m_1(\tau(t-\sigma)+\mu_1(c(a-\sigma,t-\sigma))]d\sigma} & \text{when } a \geq t, \\
  b(t)v(t-a)e^{-\int_0^a[m_1(\tau(t-\sigma)+\mu_1(c(a-\sigma,t-\sigma))]d\sigma} & \text{when } a < t,
\end{cases}
\]

\( (29) \)

\[
v(t) = v_0 e^{-\int_0^t m_2(\tau(t)) \mu_2(L_1(c))d\xi + \int_0^t u(l,\sigma)e^{-\int_\sigma^t m_2(\tau(t)) \mu_2(L_1(c))d\xi} d\sigma}.
\]

\( (30) \)

From the expressions in (29) and the known positivities, when \( a = l \), we get

\[
|u(l,t)| \leq b_m v_0 + b_m \int_0^t |u(l,\sigma)|, \text{ when } l < t,
\]

\[
|u(l,t)| \leq \|u_0\|_{L^\infty(0,l)}, \text{ when } l \geq t,
\]

and consequently,

\[
|u(l,t)| \leq b_m v_0 + \|u_0\|_{L^\infty(0,l)} + b_m \int_0^t |u(l,\sigma)| d\sigma, \quad \forall t \in [0,T].
\]

Thus, by Gronwall’s inequality, we obtain

\[
|u(l,t)| \leq (b_m v_0 + \|u_0\|_{L^\infty(0,l)})e^{b_m T}, \quad \forall t \in [0,T],
\]
which, together with the known positives and (30), implies that
\[ |v(t)| \leq v_0 + \int_0^t |u(l, \sigma)| \leq v_0 + (b_m v_0 + \|u_0\|_{L^\infty([0,1])}) e^{b_m T} t, \quad \forall t \in [0, T], \]
which implies (25) with \( \delta_2 \) given by (28).

On the other hand, again by (29) and the known positivities, we obtain
\[ |u(a, t)| \leq \|u_0\|_{L^\infty([0,1])}, \text{ when } a \geq t, \]
\[ |u(a, t)| \leq |b_m v(t - a)|, \text{ when } a < t. \]
By using (31) in this last inequality, combined with the previous one, we obtain (24) with \( \delta_1 \) given by (27).

\[ \Box \]

4. Proof of Theorem 2

We proceed similarly as we did in the proof of Proposition 5; we take \( 0 < T_1 \leq T \) and consider the complete metric space \( X_1 = L^\infty([0, T_1], L^1([0, 1])) \times C^+([0, T_1]) \times M([0, T_1]) \), where \( M([0, T_1]) = \{r \in C([0, T_1]; 0 \leq r \leq r_c)\} \), with the standard metric
\[ d_{X_1}((u_2, v_2, r_2), (u_1, v_1, r_1)) = \|u_2 - u_1\|_{L^\infty([0, T_1], L^1([0, 1]))} \]
\[ + \|v_2 - v_1\|_{C([0, T_1])} + \|r_2 - r_1\|_{C([0, T_1])}. \]
We also consider the operator defined by
\[ T_1 : X_1 \longrightarrow X_1, \]
\[ (u, v, r) \mapsto T_1(u, v, r) = (u, v, r) \]
where \((u, v, r)\) is the unique solution of system (23) given by Proposition 7; the estimates stated in this same proposition guarantee that \( T_1 \) is a well-defined.

As before, we will show that the operator \( T_1 \) has a unique fixed point for \( T_1 \) small enough. For this, besides our general hypotheses, the estimates obtained in Proposition 7 will be fundamental; also observe that \( \|u_0\|_{L^\infty([0,1])} < \delta_1 \) and \( v_0 < \delta_2 \), where \( \delta_1 \) and \( \delta_2 \) are given respectively by (27) and (28).

To show that \( T_1 \) is a strict contraction, we denote
\[ T_1(u_i, v_i, r_i) = (u_i, v_i, r_i), \quad i = 1, 2. \]

Now, from (30) and the fact that \( m_2 \) is a Lipschitz function, after some standard computations, we get
\[ |v_2(t) - v_1(t)| \]
\[ \leq (v_0 k_{m_2} + k_{m_2} \int_0^t |u_2(l, \sigma)| d\sigma) T_1 \|F_2 - F_1\|_{C([0, T_1])} + \int_0^t |u_2(l, \sigma) - u_1(l, \sigma)| d\sigma \]
\[ \leq (\delta_2 k_{m_2} + k_{m_2} \delta_1) T_1 \|F_2 - F_1\|_{C([0, T_1])} + \int_0^t |u_2(l, \sigma) - u_1(l, \sigma)| d\sigma. \]

Next, from (29) and the fact that \( m_1 \) is a Lipschitz function, again after some computations, we obtain when \( a < t \):
\[ |u_2(a, t) - u_1(a, t)| \]
\[ \leq b |v_2(t - a) - v_1(t - a)| + b_m k_{m_1} |v_1(t - a)| \int_0^a |F_2(t - \sigma) - F_1(t - \sigma)| d\sigma \]
\[ \leq b |v_2(t - a) - v_1(t - a)| + b_m k_{m_1} \delta_2 \int_0^t |F_2(\xi) - F_1(\xi)| d\xi \]
\[ \leq b |v_2(t - a) - v_1(t - a)| + b_m k_{m_1} \delta_2 T_1 \|F_2 - F_1\|_{C([0, T_1])}. \]
Similarly, when \( a \geq t \) we get
\[
|u_2(a, t) - u_1(a, t)| \leq k_{m_1} \|u_0\|_{L^\infty(0, t)} T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]}.
\] (34)

These last two estimates in particular hold for \( a = l \), and thus, when \( l < t \),
\[
|u_2(l, t) - u_1(l, t)| \leq b_m |v_2(t - l) - v_1(t - l)| + b_m k_{m_1} \delta_2 T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]}
\]
\[
\leq (b_m d_1 + b_m k_{m_1} \delta_2) T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]}
\]
\[
+ \int_0^t |u_2(l, \sigma) - u_1(l, \sigma)| d\sigma,
\] (35)

where
\[
d_1 = k_{m_2} (\delta_2 + \delta_1 T).
\]

From (35) and (34), we obtain that \( \forall t \in [0, T_1] \) there holds
\[
|u_2(l, t) - u_1(l, t)| \leq d_2 T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]} + b_m \int_0^t |u(l, \sigma) - u(l, \sigma)| d\sigma
\]
where
\[
d_2 = b_m d_1 + k_{m_1} \delta_1 + b_m k_{m_1} \delta_2.
\]

Due to Gronwall’s inequality we then get
\[
|u_2(l, t) - u_1(l, t)| \leq d_2 e^{b_m T} T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]},
\]
which, together with (32), gives for all \( t \in [0, T_1] \) that
\[
|v_2(t) - v_1(t)| \leq d_3 T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]},
\] (36)

where
\[
d_3 = d_1 + d_2 e^{b_m T} T.
\]

This last inequality, (33) and (34) then gives for all \( t \in [0, T_1] \) that
\[
|u_2(a, t) - u_1(a, t)| \leq d_4 T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]},
\] (37)

where
\[
d_4 = b_m d_3 + b_m k_{m_1} \delta_2 + k_{m_1} \delta_1.
\]

This implies in particular
\[
\|u_2 - u_1\|_{L^\infty(0, T_1, L^1(0, t))} \leq d_4 T_1 \|\mathbf{r}_2 - \mathbf{r}_1\|_{C[0, T_1]}.
\] (38)

On the other hand, from the third equation in (23), we have that \( r(t) = r_0 e^{\int_0^t (g(r(\xi)) - h(L_2(u, v))) d\xi} \). Then, by using the mean value theorem, the nonnegativity of \( h(\cdot) \), (26) and the fact that \( h \in g \) are Lipschitz functions, we get
\[
|r_2(t) - r_1(t)|
\]
\[
\leq r_0 \left( k_g \int_0^t |r_2(\xi) - r_1(\xi)| d\xi + k_h e^{\gamma_1} \int_0^t |L_2(u_2, v_2) - L_2(u_1, v_1)| d\xi \right)
\]
\[
\leq r_0 k_g \int_0^t |r_2(\xi) - r_1(\xi)| d\xi
\]
\[
+ r_0 k_h e^{\gamma_1} \int_0^t \int_0^t |u_2(\theta, \xi) - u_1(\theta, \xi)| H_1(\theta, \xi) d\theta d\xi
\]
\[
+ r_0 k_h e^{\gamma_1} \int_0^t \int_0^t |v_2(\xi) - v_1(\xi)| H_2(\theta, \xi) d\theta d\xi
\] (39)

where \( g_m \) is the natural maximum growth rate defined in (7).
By using (36) an (37) in this last inequality, recalling that \( r_0 \leq d \), we obtain

\[
|r_2(t) - r_1(t)| \leq d_5 T_1 \|\tau_2 - \tau_1\|_{C[0,T_1]} + \tilde{d}_5 \int_0^t |r_2(\xi) - r_1(\xi)|d\xi,
\]

where

\[
d_5 = r_c k_h e^{\gamma m T} (d_4 T l) \|H_1\|_{L^\infty(Q)} + d_5 \|H_2\|_{L^\infty(Q)}, \\
\tilde{d}_5 = r_0 k_g.
\]

By applying Gronwall’s inequality to the last inequality, we then get

\[
|r_2(t) - r_1(t)| \leq d_6 T_1 \|\tau_2 - \tau_1\|_{C[0,T_1]},
\]

with

\[
d_6 = d_5 e^{\tilde{d}_5 T}.
\]

From (36), (38) and (40), we obtain

\[
d_{X_1}(\mathcal{T}(\pi_2, \tau_2), \mathcal{T}(\pi_1, \tau_1)) \leq DT_1 d_{X_1}((\pi_2, \tau_2), (\pi_1, \tau_1)),
\]

where

\[
D = d_3 + d_4 l + d_6.
\]

From this last inequality, by taking \( T_1 = \min\{1/(2D), T\} \), \( \mathcal{T}_1 \) is a strict contraction with contraction constant 1/2; thus it has a unique fixed point which is a solution of (1) on \([0, T_1]\).

Moreover, we observe from the expressions of the constants appearing in the previous estimates that the constant \( D \) depends polynomially only on \( l, T, r_c, \delta_1, \delta_2, k_m, k_m, k, k_h, e^{r_0 k_g} \exp(T g_m) T, e^{b_n T}, \|H_1\|_{L^\infty(Q)}, \|H_2\|_{L^\infty(Q)} \).

Next, we repeat all the previous arguments, but now with the initial conditions \( (u(\cdot, T_1), v(T_1)) \) instead of \( (u_0(\cdot), v_0), r_0 \), with the functional space \( X_2 = L^\infty(T_1, T_2, L^1(0,l)) \times C^+[T_1, T_2] \times M[T_1, T_2] \) instead of \( X_2 \) and an operator \( \mathcal{T}_2 : X_2 \to X_2 \) defined similarly as before.

Since from the estimates in Proposition 7, we have \( u(a, T_1) \leq \delta_1, v(T_1) \leq \delta_2 \) and \( r(T_1) \leq r_c \), exactly the same computations apply in this case, and we conclude that by taking \( T_2 = \min\{T_1 + 1/(2D), T\} \) then \( \mathcal{T}_2 \) is again a strict contraction with contraction constant again equal to 1/2; this gives once more a unique fixed point which extends the previously obtained solution of (1) to the interval \([T_1, T_2]\). By proceeding successively in this way, we extend the solution to the intervals \([T_2, T_3], [T_3, T_4], \ldots\), with size 1/(2D), except the last one, and after a finite number of steps we finally obtain a unique solution (1) to the interval \([0, T]\).

Finally the estimates in (12) are proved exactly as the corresponding one is Proposition 7. The proof of Theorem 2 is then complete. □

5. Proof of Theorem 3

To prove Theorem 3, we will use the direct method of calculus of variations, that is, an argument based on minimization sequences. For this, we firstly observe from the hypotheses in (H6) that

\[
\mathcal{F} = \inf\{\mathcal{F}(c) : c \in U\} > -\infty;
\]

where

\[
\mathcal{F}(c) = \int_0^T f(t, c(t), c'(t), \mathcal{A}c(t)) dt.
\]
next, we take a minimization sequence \( \{c_n\}_{n=1}^{\infty} \subset U \); that is, one such that
\[
\lim_{n \to +\infty} F(c_n) = F.
\] (44)

We then denote by \((u_n, v_n, r_n)\) the solutions of (1) corresponding to the controls \(c_n\) given by Theorem 2.

Then, we observe that the expression of \(F\) given in (3) and the fact that \(G\) is lower bounded in particular gives that there is a positive constant \(C\), independent of \(n\) such that
\[
\|c_n\|_{L^p(\Omega)} \leq C, \quad \|c_{n,t}\|_{L^p(\Omega)} \leq C, \quad \|c_{n,a}\|_{L^p(\Omega)} \leq C, \\
\|v_n\|_{L^p(0,T)} \leq C, \quad \|r_n\|_{L^p(0,T)} \leq C.
\] (45)

We remark that we are using that \(\rho_2, \rho_3 > 0\); it is very easy to adapt the following arguments to the cases where one or both of these parameters are zero.

On the other hand, from the estimates in (45), taking in consideration how the estimates (12) obtained in Theorem 2 depend on the problem data, we obtain that there is a positive constant \(C\), independent of \(n\) such that
\[
\|u_n\|_{L^q(\Omega)} \leq C(\|T\|^{1/q}) \quad \text{for the } q \text{ given in (H6)}.
\] (47)

From the estimates in (45) and (46), by using Banach–Alaoglu–Bourbaki theorem (see for instance Brezis [30], Theorem 3.16, p. 66) and the fact that some of the involved spaces are reflexive due the condition \(1 < p_1, \tilde{p}_1, \tilde{p}_1, p_2, p_3 < +\infty\) in the hypotheses (H6), there are functions
\[
u \in L^\infty(\Omega), \quad v \in L^{p_2}(0,T) \cap L^\infty(0,T), \quad r \in L^{p_3}(0,T) \cap L^\infty(0,T),
\]
\[
c \in L^{p_1}(\Omega), \quad \tilde{c} \in \tilde{L}^{p_1}(\Omega), \quad \tilde{c} \in \tilde{L}^{p_1}(\Omega),
\]
for which hold the same estimates of (45), (46) and (47) and there are subsequences, which for simplicity of exposition, we do not relabel, such that the following convergences hold:
\[
u_n \to u \text{ weakly-* in } L^\infty(\Omega), \quad u_n \to u \text{ weakly in } L^q(\Omega),
\]
\[
v_n \to v \text{ weakly in } L^{p_2}(0,T), \quad r_n \to r \text{ weakly in } L^{p_3}(0,T).
\]
\[
c_n \to c \text{ weakly in } L^{p_1}(\Omega),
\]
\[
c_{n,t} \to \tilde{c} = c_t \text{ weakly in } \tilde{L}^{p_1}(\Omega), \quad c_{n,a} \to \tilde{c} = c_a \text{ weakly in } L^{p_1}(\Omega).
\] (48)

As stated above, \(\tilde{c} = c_t\) and \(\tilde{c} = c_a\); these results are ease consequence of the following arguments: the first of the previous convergences implies convergence in distribution, which then implies also that \(c_n,t \to c_t\) and \(c_{n,a} \to c_a\) in distribution; on the other hand, the second of the above convergences also implies that \(c_{n,t} \to \tilde{c}\) is distribution; finally, due to the uniqueness of the limits in distribution, we then obtain \(\tilde{c} = c_t\) and \(\tilde{c} = c_a\).

We also observe that, since \(Q\) has finite measure, the first and the third estimates in (45) together imply the following estimates independent of \(n\):
\[
\|c_n\|_{L^{p_1}(0,T; W^{1,q}_1(0,l))} \leq C, \quad \text{and} \quad \|c_{n,t}\|_{L^{p_1}(0,T; L^{p_1}(0,l))} \leq C.
\] (49)

where \(q_1 = \min\{p_1, \tilde{p}_1\}\). Due to (10), \(W^{1,q}_1(0,l)\) is compactly imbedded in \(L^{p_1}(0,l)\), by using the variant of the Aubin–Lions lemma given in Simon [32], Corollary 4, p. 85, along subsequences, which again we do not relabel, we have
\[
c_n \to c \text{ strongly in } L^{p_1}(0,T; L^{p_1}(0,l)) \subset L^{s}(0,T; L^{s}(0,l)) = L^{s}(Q),
\]
where \(s = \min\{q_1, \tilde{p}_1\}\). (50)
Moreover, from hypothesis (H5) we know that the set of admissible controls is $U$ a convex set which is closed in the strong topology of $L^{p_1}(Q)$ and thus also closed in the weak topology of $f L^{p_1}(Q)$, see Brezis [30], Theorem 3.7, p. 60. Thus, we get

$$c \in U. \quad (51)$$

Next, we observe that, due to the conditions (9) given in (H6), it is easy to prove that the first term in the expression of (3) is continuous in the strong (norm) topology of $L^q(Q)$ ($q$ is given in (H6)). Since by the other hypotheses in (H6), the first term in (3) is also a convex functional, it is lower semi-continuous in the weak topology of $L^q(Q)$ (see for instance Brezis [30], Corollary 3.9, Remark 6, p. 61).

We similarly conclude that the other terms in (3) are also lower semi-continuous in the respective weak topologies.

Due to such lower semi-continuities and (44), we obtain

$$\mathcal{F}(c) \leq \lim_{n \to +\infty} \mathcal{F}(c_n) = \mathcal{F}. \quad (52)$$

From this, (51) and (43), we then obtain that

$$c \in U \text{ and } \mathcal{F}(c) = \mathcal{F} = \inf \{ \mathcal{F}(c) : c \in U \}. \quad (53)$$

To conclude that such $c$ is in fact an optimal control, the only thing that is left to prove is that $(u,v,r)$ is the solution of (11) associated to $c$.

To prove this, we start with the fact that for each $n$ we have

$$\begin{align*}
    u_n(a,t) &= \begin{cases} 
    u_0(a - t)e^{-\int_0^a [m_1(r_n(t-\sigma)+\mu_2(c_n(\sigma,t-\sigma))]d\sigma} & \text{for } a \geq t, \\
    b(t)v_n(t-a)e^{-\int_0^a [m_1(r_n(t-\sigma))+\mu_1(c_n(\sigma,t-\sigma))]d\sigma} & \text{for } a < t,
    \end{cases} \\
    v_n'(t) + m_2(r_n(t))v_n(t) + \mu_2(L_1(c_n)(t))v_n(t) &= u_n(l,t), \\
    r_n'(t) - [g(r_n(t)) - h(L_2(u_n,v_n)(t))]r_n &= 0, \\
    v_n(0) &= \psi_0, \\
    r_n(0) &= \rho_0,
\end{align*} \quad (54)$$

and then we will try to pass the limit as $n \to +\infty$ in the above equations to get that $(u,v,r)$ and $c$ satisfies (11).

However, due to the nonlinearities, the weak converges in (48) are not enough to accomplish this task. Certain strong converges are required, and to obtain them, we must first look for estimates for the time derivatives.

For this, we firstly observe that, since $\mu_2$ is a nonnegative Lipschitz function, for any $z \in \mathbb{R}$ we have $|\mu_2(z)| \leq \mu_2(0) + k_{\mu_2}|z|$. Thus, by using Hölder’s inequality, the expression for $L_1$ given in (6) and the first estimate in (45), after some computations, we get the following estimate independent of $n$:

$$\|\mu_2(L_1(c_n))\|_{L^{p_1}(0,T)} \leq T^{1/p_1} \mu_2(0) + k_{\mu_2}\|H_0\|_{L^{\infty}(Q)}^{1-(1/p_1)}\|c_n\|_{L^{p_1}(Q)} \quad (55)$$

Now, from the second equation in (53), we have for each $n$ that

$$v_n'(t) = -m_2(r_n(t))v_n(t) - \mu_2(L_1(c_n))v_n(t) + u_n(l,t),$$
which, together with (46) and (54), gives the following estimates also independent of \(n\):

\[
\|v'_n\|_{L^p(0,T)} \leq (M_2T^{1/p_1} + \tilde{C})\|v_n\|_{L^\infty(0,T)} + \|u_n\|_{L^\infty(Q)}
\]

where \(M_2 = \max\{m_2(z) : 0 \leq z \leq r_c\}\).

This last estimate implies that we can extract a subsequence (which we do not relabel) such that

\[
v'_n \to v' \quad \text{weakly in } L^{p_1}(0,T).
\]

(The fact that the limit of this subsequence is in fact \(v'\) is proved by using arguments with distribution as we previously did after (48).)

Next, we will obtain an estimate for the time derivative of \(r(t)\). For this, we firstly observe that

\[
0 \leq L_2(u_n, v_n)(t) \leq (\|H_1\|_{L^\infty(Q)} + \|H_2\|_{L^\infty(Q)})lC \equiv C_2,
\]

and thus we have the following estimate for all \(n\):

\[
0 \leq h(L_2(u_n, v_n)(t)) \leq h_m = \max\{h(z) : 0 \leq z \leq C_2\}.
\]

Now, from the third equation in (53), we have for each \(n\) that

\[
r'_n(t) = [g(r_n(t)) - h(L_2(u_n, v_n)(t))]r_n(t),
\]

which, due to the last estimate and (7), gives the following estimate independent of \(n\):

\[
\|r'_n\|_{L^\infty(0,T)} = [g_m + h_m]C \equiv C_3.
\]

From this, we can extract a subsequence (which for simplicity of notations we do not relabel), such that

\[
r'_n \to r' \quad \text{weakly-* in } L^\infty(0,T).
\]

(The fact that the limit of this subsequence is in fact \(r'\) is proved by using arguments with distribution as we previously did after (48).)

Moreover, estimates (55) and (57) imply respectively that the sequences \(v_n\) and \(r_n\) are equicontinuous on the time interval \([0,T]\). These facts together with the \(L^\infty(0,T)\) estimates for \(v_n\) and \(r_n\) in (46) and Arzelà–Ascoli theorem, imply that both \(v(\cdot)\) and \(r(\cdot)\) are absolutely continuous and also that the following uniform convergence of subsequences (which again do not relabel) holds:

\[
v_n \to v \quad \text{and} \quad r_n \to r \quad \text{in } C[0,T].
\]

To be able to pass to the limit in (53), we need to prove some more estimates.

For this, we recall that the mean value theorem, for \(z_1, z_2 \leq 0\) implies that \(|\exp z_1 - \exp z_2| \leq |z_1 - z_2|\).

Then, for \(t < a\) we can write

\[
\begin{align*}
&\left|e^{-\int_0^t \mu_1(c_n(a-\sigma,t-\sigma))d\sigma} - e^{-\int_0^t \mu_1(c(a-\sigma,t-\sigma))d\sigma}\right| \\
&\leq \left|\int_0^t \mu_1(c_n(a-\sigma,t-\sigma))d\sigma - \int_0^t \mu_1(c(a-\sigma,t-\sigma))d\sigma\right| \\
&\leq k_{\mu_1} \int_0^t |c_n(a-\sigma,t-\sigma) - c(a-\sigma,t-\sigma)|d\sigma \\
&\leq k_{\mu_1} \chi_{\mathcal{Q}_1}(a,t) \int_0^T |c_n^{\text{ext}}(a-\sigma,t-\sigma) - c^{\text{ext}}(a-\sigma,t-\sigma)|d\sigma,
\end{align*}
\]
where \( Q_1 = \{(a, t) \in Q : a \geq t\} \), \( \chi_{Q_1} \) denotes the characteristic function of \( Q_1 \) and \( c_n^{ext} \) and \( c^{ext} \) are the extensions respectively of \( c_n \) and \( c \) to \((-l, l) \times (-T, T)\) obtained by reflexion on the \( t \)-axis and \( \alpha \)-axis.

Now, by using Hölder’s inequality and Fubini’s theorem, we can estimate as

\[
\|e^{-\int_0^t \mu_1(c_n(a-\sigma, t-\sigma) d\sigma)} - e^{-\int_0^t \mu_1(c(a-\sigma, t-\sigma) d\sigma)}\|^s_{L^s(Q)} \\
\leq k_1^s \int_0^T \int_0^T \int_0^1 \int_0^{1-\sigma} \|c_n^{ext}(\xi_1, \xi_2) - c^{ext}(\xi_1, \xi_2)\|^s d\xi_1 d\xi_2 d\sigma \\
\leq k_1^s T^{1-s} \int_0^T \int_0^T \int_0^1 \int_0^{1-\sigma} \|c_n^{ext}(\xi_1, \xi_2) - c^{ext}(\xi_1, \xi_2)\|^s d\xi_1 d\xi_2 d\sigma \\
= k_1^s T^{1-s} \int_0^T \int_0^T \int_0^1 \int_0^{1-\sigma} \|c_n^{ext}(\xi_1, \xi_2) - c^{ext}(\xi_1, \xi_2)\|^s d\xi_1 d\xi_2 d\sigma.
\]

Next, by a change of variables, we get

\[
\|e^{-\int_0^t \mu_1(c_n(a-\sigma, t-\sigma) d\sigma)} - e^{-\int_0^t \mu_1(c(a-\sigma, t-\sigma) d\sigma)}\|^s_{L^s(Q)} \\
\leq k_1^s T^{1-s} \int_0^T \int_0^T \int_0^1 \int_0^{1-\sigma} \|c_n^{ext}(\xi_1, \xi_2) - c^{ext}(\xi_1, \xi_2)\|^s d\xi_1 d\xi_2 d\sigma \\
\leq k_1^s T^{1-s} \int_0^T \int_0^T \int_0^1 \int_0^{1-\sigma} \|c_n^{ext}(\xi_1, \xi_2) - c^{ext}(\xi_1, \xi_2)\|^s d\xi_1 d\xi_2 d\sigma \\
= k_1^s T^{1-s} \int_0^T \int_0^T \int_0^1 \int_0^{1-\sigma} \|c_n^{ext}(\xi_1, \xi_2) - c^{ext}(\xi_1, \xi_2)\|^s d\xi_1 d\xi_2 d\sigma.
\]

In short, we obtain

\[
\|e^{-\int_0^t \mu_1(c_n(a-\sigma, t-\sigma) d\sigma)} - e^{-\int_0^t \mu_1(c(a-\sigma, t-\sigma) d\sigma)}\|^s_{L^s(Q)} \\
\leq k_1^s T^{1-s} \|c_n - c\|^s_{L^s(Q)}.
\]

Similarly working, we also obtain for \( a < t \)

\[
\|e^{-\int_0^a \mu_1(c_n(a-\sigma) d\sigma)} - e^{-\int_0^a \mu_1(c(a-\sigma) d\sigma)}\|^s_{L^s(Q)} \\
\leq k_1^s 4T^{1-s} \|c_n - c\|^s_{L^s(Q)}.
\]

We observe that, due to (50), the right-hand sides of (60) and (61) go to zero as \( n \to +\infty \).

In an easier way, we also get

\[
\|e^{-\int_0^t \mu_1(m_n(r(t-\sigma)) d\sigma)} - e^{-\int_0^t \mu_1(m(r(t-\sigma)) d\sigma)}\|^s_{L^s(Q)} \\
\leq k_1^s T^{s+1} \|r_n - r\|^s_{C[0,T]},
\]

which goes to zero as \( n \to +\infty \) due to (59).

From (60), (61), (62) and also the uniform convergence for \( u_n \) given in (59), we can pass to the limit as \( n \to +\infty \) in the first equation in (53) to get that the first equation in (11) holds in \( L^s(Q) \).

To pass to the limit in the second equation of (53), for \( s \) as in (50) and the expression of \( L_1 \) given in (6), we observe that

\[
\|\mu_2(L_1(c_n)) - \mu_2(L_1(c))\|^s_{L^s(Q)} \\
\leq k_2^s \int_0^T \int_0^1 |c_n(a,t) - c(a,t)| H_0(a,t) da dt \\
\leq k_2^s \|H_0\|_{L^\infty(Q)} T^{s-1} \|c_n - c\|^s_{L^s(Q)},
\]

which goes to zero as \( n \to +\infty \) due to (50).
By using (56), the uniform convergence of \( r_n \) and \( v_n \) given in (59), the fact that \( m_2(\cdot) \) is a Lipschitz function, and (63), we can pass to the limit as \( n \to +\infty \) in the second equation of (53) to get that the second equation in (11) holds in \( L^2(0,T) \), where \( \bar{s} = \min\{p_1, s\} = \min\{p_1, \min\{q_1, \bar{p}_1\}\} \); since we know that \( v(\cdot) \) is absolutely continuous, the second equation in (11) holds in the sense of Carathéodory.

Next, to pass to the limit in the third equation of (53), from the expression of \( L_2 \) given in (8), we observe that

\[
\begin{align*}
    h(L_2(u_n, v_n)(t)) - h(L_2(u, v)(t)) &= c_0 \int_0^t (u_n(a, t) - u(a, t)) H_1(a, t) da + c_0 \int_0^t (v_n(t) - v(t)) H_2(a, t) da. \\
\end{align*}
\]

(64)

The last term in the previous right-hand side can be estimated as

\[
\|c_0 \int_0^t (v_n(t) - v(t)) H_2(a, t) da\|_{L^\infty(0,T)} \leq c_0 \|H_0\|_{L^\infty(Q)} \|v_n - v\|_{C[0,T]},
\]

which goes to zero as \( n \to +\infty \) due to (59).

As for the first term in the right-hand side of (64), we observe that for any \( \phi \in L^1(0, T) \) there holds:

\[
\begin{align*}
    \int_0^T c_0 \int_0^t (u_n(a, t) - u(a, t)) H_1(a, t) da \phi(t) dt \\
    \int_0^T \int_0^t (u_n(a, t) - u(a, t)) c_0 H_1(a, t) \phi(t) da dt &\to 0,
\end{align*}
\]

since \( c_0 H_1(\cdot, \cdot) \phi(\cdot) \in L^1(Q) \) and \( u_n \) converges weakly-* to \( u \) in \( L^\infty(Q) \) by (48).

From (64), and the two last observations, we conclude that

\[
\begin{align*}
    h(L_2(u_n, v_n)) &\to h(L_2(u, v)) \text{ weakly in } L^\infty(0,T).
\end{align*}
\]

(65)

From (58), the fact that \( g(\cdot) \) is a continuous function, the uniform convergence of \( r_n \) to \( r \) given in (59) and (65), we can pass to the limit as \( n \to +\infty \) in the third equation of (53) to get that the third equation in (11) holds in \( L^\infty(0,T) \); since we know that \( v(\cdot) \) is absolutely continuous, the third equation in (11) holds in the sense of Carathéodory.

The initial conditions \( v(0) = v_0 \) and \( r(0) = r_0 \) are consequences of the uniform convergences in (59) and the facts that \( v_n(0) = v_0 \) and \( r_n(0) = r_0 \).

Therefore, \((u, v, r)\) together with \( c \) is the required solution of (11), and we conclude that \( c \) is in fact an optimal control for the problem. \( \square \).

6. Optimality conditions

We stress that in the present section we assume the conditions given in the statement of Theorem 4. In particular, these conditions require that \( \mu_1(\cdot) \) and \( \mu_2(\cdot) \) are bounded functions; however, due to the \( L^\infty \)-estimates (12) for \( u, v \) and \( r \), we can also assume that the coefficients \( m_1(\cdot), m_2(\cdot), g(\cdot) \) and \( h(\cdot) \) are bounded; in fact, it is enough to modify them outside suitable bounded intervals, which throughout this section we suppose is done.

We start by fixing an admissible optimal control \( c^* \) of (2) with corresponding optimal states \( u^*, v^* \) and \( r^* \). For each \( 0 < \epsilon \leq 1 \), we consider an approximate pure optimization problem obtained by a ‘least-square’ penalization of the dynamic equations and localization.

We start by recalling that we denote \( Q = (0, l) \times (0, T) \) and define

\[
V = \{ u \in L^2(Q) : u_t + u_a \in L^2(Q) \}
\]
which, endowed with the usual operations on functions and norm defined by \( \| (u, v) \|_V = (\| u \|^2_{L^2(Q)} + \| u_t + u_u \|^2_{L^2(Q)})^{1/2} \), is a Hilbert space. Moreover, elements of \( V \) admit traces at the boundaries of \( Q \); for instance, it is fair to take the trace \( u(0, \cdot) \) and \( \| u(0, \cdot) \|_{L^2(0,T)} \leq C \| u \|_V \), with a constant \( C > 0 \) independent of \( u \in V \). We also consider the following closed affine subspace:

\[
V_{u_0} = \{ u \in V : u(\cdot, 0) = u_0(\cdot) \}.
\]

We also consider the following affine subspaces of \( H^1(0, T) \):

\[
V_{v_0} = \{ v \in H^1(0, T) : v(0) = v_0 \} \quad \text{and} \quad V_{r_0} = \{ r \in H^1(0, T) : r(0) = r_0 \}.
\]

Then, we define the set \( \mathcal{V} \) as follows.

\[
\mathcal{V} = U \times V_{u_0} \times V_{v_0} \times V_{r_0}.
\]  

(66)

Next, we consider:

**Approximate problem:** find \((c^*_\epsilon, u^*_\epsilon, v^*_\epsilon, r^*_\epsilon)\) such that

\[
J_\epsilon(c^*_\epsilon, u^*_\epsilon, v^*_\epsilon, r^*_\epsilon) = \min \{ J_\epsilon(c, u, v, r) : (c, u, v, r) \in \mathcal{V} \text{ with } J_\epsilon(c, u, v, r) < +\infty \},
\]

(67)

where the functional \( J_\epsilon(c, u, v, r) \) to be minimized is given by

\[
J_\epsilon(c, u, v, r) = J(c, u, v, r) + \frac{1}{2\epsilon} \| R_u \|^2_{L^2(Q)} + \frac{1}{2\epsilon} \| R_v \|^2_{L^2(0,T)} \nonumber\]

\[
+ \frac{1}{2\epsilon} \| R_r \|^2_{L^2(0,T)} + \frac{1}{\epsilon} \| R_{bc} \|^2_{L^2(0,T)} \nonumber\]

\[
+ \frac{1}{2} \| c - c^* \|^2_{L^2(Q)} + \frac{1}{2} \| u - u^* \|^2_{L^2(Q)} \nonumber\]

\[
+ \frac{1}{2} \| v - v^* \|^2_{L^2(0,T)} + \frac{1}{2} \| r - r^* \|^2_{L^2(0,T)}.
\]  

(68)

Here, \( J(c, u, v, r) \) has exactly the expression of (3) with \( p_1 = \tilde{p}_1 = \bar{p}_2 = p_2 = p_3 = 2 \), that is,

\[
J(c, u, v, r) = \rho_0 \int_0^T \int_0^l G(a, u(a, t)) \, dt \, dt + \rho_1 \int_0^T \int_0^l |c(a, t)|^2 \, dt \, dt
\]

\[
+ \tilde{p}_1 \int_0^T \int_0^l |c_t(a, t)|^2 \, dt \, dt + \bar{p}_1 \int_0^T \int_0^l |c_u(a, t)|^2 \, dt \, dt
\]

\[
+ \rho_2 \int_0^T |v(t)|^2 \, dt + \rho_3 \int_0^T |r(t)|^2 \, dt.
\]

In the second line of (68) has the terms penalizing the residuals of each one of the dynamical equations and boundary condition, that is, the residuals \( R_u, R_v, R_r \) and \( R_{bc} \) are defined by

\[
\begin{align*}
&u_t + u_u + m_1(r)u + \mu_1(c)u = R_u, \\
v' + m_2(r)v + \mu_2(L_1(c))v - u(l, \cdot) = R_v, \\
r' - [g(r) - h(L_2(u, v))]|r = R_r, \\
&u(0, \cdot) - b(\cdot)v(\cdot) = R_{bc}, \\
&u(a, 0) = u_0(a), \\
v(0) = v_0, \\
r(0) = r_0.
\end{align*}
\]  

(69)
In the above equations, we understand that the operator $L_2$ still has the expression given in (8), but it is now understood to act as $L_2 : L^2(Q) \times C[0,T] \rightarrow L^\infty(0,T)$.

We remark that the terms in the third and fourth lines in (68) are penalization terms for localization near $(e^*, u^*, v^*, r^*)$.

Concerning the previous optimization problem, we have the following results:

**Proposition 8.** For each $\epsilon > 0$, problem (67) has a solution $(c^*_\epsilon, u^*_\epsilon, v^*_\epsilon, r^*_\epsilon)$.

**Proof.** We start by taking a minimizing sequence $\{((c_{\epsilon}^{(n)}), u_{\epsilon}^{(n)}, v_{\epsilon}^{(n)}, r_{\epsilon}^{(n)})\}_{n=1}^{\infty}$ associated to $J_\epsilon$. Then, by definition of $J_\epsilon$ we immediately have a uniform estimate for the corresponding residuals:

$$\frac{1}{2\epsilon} \left\| R_{u}^{(n)} \right\|_{L^2(Q)}^2 + \frac{1}{2\epsilon} \left\| R_{v}^{(n)} \right\|_{L^2(0,T)}^2 + \frac{1}{2\epsilon} \left\| R_{r}^{(n)} \right\|_{L^2(0,T)}^2 + \frac{1}{2\epsilon} \left\| R_{bc}^{(n)} \right\|_{L^2(0,T)}^2 \leq C.$$

By using this uniform estimate and proceeding similarly as in the proof of Theorem 3 (see Section 5), we obtain uniform estimates for $c_{\epsilon}^{(n)}$, $u_{\epsilon}^{(n)}$, $v_{\epsilon}^{(n)}$, $r_{\epsilon}^{(n)}$ in suitable norms which allow us to show that there is $(c^*_\epsilon, u^*_\epsilon, v^*_\epsilon, r^*_\epsilon)$ and subsequence such that $(c_{\epsilon}^{(n)}, u_{\epsilon}^{(n)}, v_{\epsilon}^{(n)}, r_{\epsilon}^{(n)}) \rightarrow (c^*_\epsilon, u^*_\epsilon, v^*_\epsilon, r^*_\epsilon)$ in senses similar to the ones in the proof of Theorem 3. By proceeding as before, one then can take to the limit as $n \rightarrow \infty$ and obtain, as before, that $(c^*_\epsilon, u^*_\epsilon, v^*_\epsilon, r^*_\epsilon)$ is an optimal solution of problem (67). We omit the details since all the computations are very similar to the ones in Section 5. □

**Proposition 9.** As $\epsilon \rightarrow 0+$, we have

$$
\begin{align*}
&c^*_\epsilon \rightarrow c^* \text{ strongly in } L^2(Q), \quad c_{\epsilon,t}^* \rightarrow c_t^*, \quad c_{\epsilon,a}^* \rightarrow c_a^* \text{ weakly in } L^2(Q); \\
u^*_\epsilon \rightarrow u^* \text{ weakly in } L^2(Q); \\
v^*_\epsilon \rightarrow v^* \text{ strongly in } C[0,T], \quad (v^*_\epsilon)' \rightarrow (v')' \text{ weakly in } L^2(0,T); \\
r^*_\epsilon \rightarrow r^* \text{ strongly in } C[0,T], \quad (r^*_\epsilon)' \rightarrow (r')' \text{ weakly in } L^2(0,T).
\end{align*}
\] (70)

Moreover, we obtain

$$
J_\epsilon(c_{\epsilon}^*, u_{\epsilon}^*, v_{\epsilon}^*, r_{\epsilon}^*) \rightarrow J(c^*, u^*, v^*, r^*). \quad (71)
\]

**Proof.** We observe that $(c^*, u^*, v^*, r^*) \in V$ (see (66)), and $J_\epsilon(c^*, u^*, v^*, r^*) = J(c^*, u^*, v^*, r^*) < +\infty$; thus, since $(c_{\epsilon}^*, u_{\epsilon}^*, v_{\epsilon}^*, r_{\epsilon}^*)$ is a minimum point of $J_\epsilon$, we have

$$
J_\epsilon(c_{\epsilon}^*, u_{\epsilon}^*, v_{\epsilon}^*, r_{\epsilon}^*) \leq J_\epsilon(c_{\epsilon}^{(n)}, u_{\epsilon}^{(n)}, v_{\epsilon}^{(n)}, r_{\epsilon}^{(n)}) \leq J(c^*, u^*, v^*, r^*), \quad \forall 0 < \epsilon \leq 1.
\] (72)

The next arguments are rather easy adaptations of the arguments and computations presented in the proof of Theorem 3 (Section 5) by taking in consideration the residual terms present in the right-hand sides of the equations in (69). We just sketch the arguments, showing the differences of the functional spaces.

Since $G$ is bounded below, from the second inequality in (72), in particular we obtain there is a positive constant $C$, independent of $\epsilon$ such that

$$
\begin{align*}
\|c_{\epsilon}^*\|_{L^2(Q)} \leq C, & \quad \|c_{\epsilon,t}^*\|_{L^2(Q)} \leq C, & \quad \|c_{\epsilon,a}^*\|_{L^2(Q)} \leq C, \\
\|u_{\epsilon}^*\|_{L^2(Q)} \leq C, & \quad \|v_{\epsilon}^*\|_{L^2(0,T)} \leq C, & \quad \|r_{\epsilon}^*\|_{L^2(0,T)} \leq C,
\end{align*}
\] (73)

and also

$$
\|R_{u}^c\|_{L^2(Q)}^2 + \|R_{v}^c\|_{L^2(0,T)}^2 + \|R_{r}^c\|_{L^2(0,T)}^2 \leq 2\epsilon J(c^*, u^*, v^*, r^*) \leq 2J(c^*, u^*, v^*, r^*), \quad (74)
\] where $R_{u}^c$, $R_{v}^c$ and $R_{r}^c$ are the residuals obtained from (69) with $(c_{\epsilon}^{(n)}, u_{\epsilon}^{(n)}, v_{\epsilon}^{(n)}, r_{\epsilon}^{(n)})$.  

From the previous results, there are \((\tilde{c}, \tilde{u}, \tilde{v}, \tilde{r})\) and subsequences, which for simplicity of exposition, we do not relabel, such that as \(\epsilon \to +\) the following convergences hold:

\[
\begin{align*}
  u^*_\epsilon & \to \tilde{u} \quad \text{weakly in } L^2(Q), \quad v^*_\epsilon \to \tilde{v} \quad \text{weakly in } L^2(0,T), \\
  r^*_\epsilon & \to \tilde{r} \quad \text{weakly in } L^2(0,T), \quad c^*_\epsilon \to \tilde{c} \quad \text{weakly in } L^2(Q), \\
  c^*_{\epsilon,t} & \to \tilde{c}_t \quad \text{weakly in } L^2(Q), \quad c_{\epsilon,a} \to \tilde{c}_a \quad \text{weakly in } L^2(Q).
\end{align*}
\]

Moreover, arguing exactly as it was done to obtain (50), we obtain

\[c^*_\epsilon \to \tilde{c} \quad \text{strongly in } L^2(Q).\] (76)

Next, by working with the second and third equations in (69) as it was done to obtain respectively (55) and (57), we conclude that

\[
\begin{align*}
  (v^*_\epsilon)' & \to \tilde{v}' \quad \text{and} \quad (r^*_\epsilon)' \to \tilde{r}' \quad \text{weakly in } L^2(0,T), \\
  v^*_\epsilon & \to \tilde{v} \quad \text{and} \quad r^*_\epsilon \to \tilde{r} \quad \text{strongly in } C[0,T],
\end{align*}
\]

Moreover, from these convergences and (74), which implies that the residuals go to zero, we conclude that \(\tilde{c} \in U\) and \((\tilde{c}, \tilde{u}, \tilde{v}, \tilde{r})\) satisfies (1).

Next, we consider the following modification of the functional \(J\)

\[
\tilde{J}(c,u,v,r) = J(c,u,v,r) + \frac{1}{2} \|c - c^*\|_{L^2(Q)}^2 + \frac{1}{2} \|u - u^*\|_{L^2(Q)}^2 + \frac{1}{2} \|v - v^*\|_{L^2(0,T)}^2 + \frac{1}{2} \|r - r^*\|_{L^2(0,T)}^2,
\]

(77)

and consider the following minimization problem:

\[\tilde{J}(c^*,u^*,v^*,r^*) = \min \{ \tilde{J}(c,u,v,r) : c \in U, (c,u,v,r) \text{ satisfies (1)} \}.\]

We stress that the minimal point \((c^*,u^*,v^*,r^*)\) we are considering is trivially also a minimal point of the just defined minimization problem, and in fact, due to the extra added terms to \(\tilde{J}\), \((c^*,u^*,v^*,r^*)\) is its unique minimal point.

Next, from the previously obtained convergences, the fact that \(J\) is a lower semicontinuous functional and also (72), we conclude that

\[
\begin{align*}
  \tilde{J}(\tilde{c},\tilde{u},\tilde{v},\tilde{r}) & \leq \liminf_{\epsilon \to 0^+} \tilde{J}(c^*_\epsilon,u^*_\epsilon,v^*_\epsilon,r^*_\epsilon) \leq \liminf_{\epsilon \to 0^+} J(c^*_\epsilon,u^*_\epsilon,v^*_\epsilon,r^*_\epsilon) \\
  & \leq \limsup_{\epsilon \to 0^+} J(c^*_\epsilon,u^*_\epsilon,v^*_\epsilon,r^*_\epsilon) \leq J(c^*,u^*,v^*,r^*) = \tilde{J}(c^*,u^*,v^*,r^*).
\end{align*}
\]

Since we already know that \(\tilde{c} \in U\) and \((\tilde{c}, \tilde{u}, \tilde{v}, \tilde{r})\) satisfies (1) and that \((c^*,u^*,v^*,r^*)\) is the unique minimal point of \(\tilde{J}\), from the last sequence of inequalities we conclude that \((\tilde{c}, \tilde{u}, \tilde{v}, \tilde{r}) = (c^*,u^*,v^*,r^*)\); thus, the convergences stated in the proposition hold. Moreover, the last sequence of inequalities also imply (71).

\[\square\]

**Proposition 10.** Under the conditions of Theorem 4, associated to any optimal solution \((c^*_\epsilon,u^*_\epsilon,v^*_\epsilon,r^*_\epsilon)\) of problem (67), there are adjoint (co-state) variables \((\lambda^1_\epsilon, \lambda^2_\epsilon, \lambda^3_\epsilon, \lambda^4_\epsilon)\) satisfying the following differential
equations

\[-[(\lambda_1)_{t} + (\lambda_1)_a] + \lambda_1^I(m_1(r_0) + \mu_1(c_0)) - \lambda_3^I h'(L_2(u_0^*, v_0^*))r_0 H_1 + \rho_0 G(\cdot, u_0^*)G_u(\cdot, u_0^*) + (u_0^* - u_0^*) = 0,\]

\[-(\lambda_2^I)' + \lambda_2^I(m_2(r_0^*) + \mu_2(L_1(c_0^*))) - \lambda_3^I h'(L_2(u_0^*, v_0^*))r_0^* (\int_0^1 H_2(a, t)da) + b \lambda_4^I + (2\rho_2 v_0^* + v_0^* - v_0^*) = 0,\]

\[-(\lambda_3^I)' + (\int_0^1 \lambda_3^I u_0^* da)m_1'(r_0^*) + \lambda_2^I m_2'(r_0^*)v_0^* + \lambda_3^I [-g'(r_0^*)r_0^* - g(r_0^*) + h(L_2(u_0^*, v_0^*))] + (2\rho_3 r_0^* + r_0^* - r_0^*),\]

\[\lambda_4^I(\cdot) - \lambda_1^I(0, \cdot) = 0,\]

together with the following final and boundary conditions

\[\lambda_1^I(\cdot, T) = 0, \quad \lambda_2^I(T) = 0, \quad \lambda_3^I(T) = 0, \quad \lambda_4^I(l, \cdot) - \lambda_2^I(\cdot) = 0,\]

and also the following differential inequality

\[2\rho_1 \int_Q c_0^* (\dot{c} - c_0^*) d\mu + 2\rho_1 \int_Q c_{0,t}^* (\dot{c}_t - c_{0,t}^*) d\mu + 2\rho_1 \int_Q c_{0,a}^* (\dot{c}_a - c_{0,a}^*) d\mu + \int_Q \lambda_1^I \mu_1^I(c_0^*) u_0^* (\dot{c} - c_0^*) d\mu + \int_0^T (c_0^* - c_0^*) (\dot{c} - c_0^*) d\mu \geq 0.\]

**Proof.**

For any \((\dot{c}, \dot{u}, \dot{v}, \dot{r}) \in V\) consider the real function \(F(z_1, z_2, z_3, z_4)\) defined for real numbers \(z_1, z_2, z_3\) and \(z_4\) by

\[F(z_1, z_2, z_3, z_4) = J_\epsilon(c_0^* + z_1(\dot{c} - c_0^*), u_0^* + z_2(\dot{u} - u_0^*), v_0^* + z_3(\dot{v} - v_0^*), r_0^* + z_4(\dot{r} - r_0^*)).\]

By observing that \((z_1, z_2, z_3, z_4) = (0, 0, 0, 0)\) is the minimum point of \(F\) and that \(\dot{u}, \dot{v}, \dot{r}\) may vary in affine subspaces while \(\dot{c}\) may vary just in a convex set, we have the following conditions for the partial Gateaux derivatives of \(F\) at \((z_1, z_2, z_3, z_4) = (0, 0, 0, 0)\):

\[D_{z_1} F((0, 0, 0, 0)) \geq 0, \quad D_{z_2} F((0, 0, 0, 0)) = 0, \quad D_{z_3} F((0, 0, 0, 0)) = 0, \quad D_{z_4} F((0, 0, 0, 0)) = 0.\]

Computing the previous derivatives, we obtain

\[D_{z_1} F((0, 0, 0, 0)) = 2\rho_1 \int_Q c_0^*(\dot{c} - c_0^*) d\mu + 2\rho_1 \int_Q c_{0,t}^*(\dot{c}_t - c_{0,t}^*) d\mu + 2\rho_1 \int_Q c_{0,a}^*(\dot{c}_a - c_{0,a}^*) d\mu + \frac{1}{\epsilon} \int_Q R_{u,z}^c d\mu + \frac{1}{\epsilon} \int_0^T R_{v,\epsilon}^c d\mu + \frac{1}{\epsilon} \int_0^T R_{r,\epsilon}^c d\mu + \frac{1}{\epsilon} \int_0^T R_{bc}^c d\mu \geq 0.\]
where, as before, $R^i_{u^i}$, $R^i_{v^i}$, $R^i_{r^i}$ and $R^i_{bc}$ are the residuals obtained from (69) with $(c^i_{u^i}, u^i_{c^i}, v^i_{c^i}, r^i_{c^i})$. Similarly, $R^i_{u_{zi^i}}$, $R^i_{v_{zi^i}}$, $R^i_{r_{zi^i}}$ and $R^i_{bc_{zi^i}}$, with $i = 1, 2, 3$, are the derivatives of such residuals. By defining

$$
\lambda^{(1)}_\epsilon(t) = \frac{1}{\epsilon} R^v_{u}(a, t), \quad \lambda^{(2)}_\epsilon(t) = \frac{1}{\epsilon} R^v_{v}(t), \quad \lambda^{(3)}_\epsilon(t) = \frac{1}{\epsilon} R^v_{r}(t), \quad \lambda^{(4)}_\epsilon(t) = \frac{1}{\epsilon} R^v_{bc}(t),
$$

after some direct but tedious computations, with the help of integration by parts, we then obtain the stated optimality conditions (78)–(80).

**Proposition 11.** There are $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$ and subsequences (we do not relabel them) such that as $\epsilon \to 0+$, we have

$$
\lambda^{(1)}_\epsilon \to \lambda^{(1)} \text{ weakly in } L^2(Q), \quad \lambda^{(2)}_\epsilon \to \lambda^{(2)} \text{ strongly in } C[0,T], \\
\lambda^{(3)}_\epsilon \to \lambda^{(3)} \text{ strongly in } C[0,T], \quad \lambda^{(4)}_\epsilon \to \lambda^{(4)} \text{ weakly in } L^2(0,T), \\
(\lambda^{(2)}_\epsilon)' \to (\lambda^{(2)}')' \text{ weakly in } L^2(0,T), \\
(\lambda^{(3)}_\epsilon)' \to (\lambda^{(3)}')' \text{ weakly in } L^2(0,T), \\
-\lambda^{(1)}_{c,\epsilon} + \lambda^{(1)}_\epsilon \to -\lambda^{(1)} + \lambda^{(1)} \text{ weakly in } L^2(Q), \\
\lambda^{(1)}_\epsilon(0,\cdot) \to \lambda^{(1)}(0,\cdot) \text{ strongly in } L^2(0,T).
$$

**Proof.** We firstly need to obtain certain estimates. We start by observing that Eqs. (78) are linear in $\lambda^{(1)}_\epsilon$, $\lambda^{(2)}_\epsilon$, $\lambda^{(3)}_\epsilon$ and $\lambda^{(4)}_\epsilon$. Next we make the change of variables $\hat{\lambda}^{(1)}_\epsilon(t) = \lambda^{(1)}_\epsilon(T - t)$, for $t \in [0,T],$
\( \dot{\lambda}_e(2)(t) = \lambda_e(2)(T - t) \), for \( t \in [0, T] \), \( \dot{\lambda}_e(3)(t) = \lambda_e(3)(T - t) \), for \( t \in [0, T] \), \( \dot{\lambda}_e(4)(t) = \lambda_e(4)(T - t) \), for \( t \in [0, T] \) and rewrite Eqs. (78) and (79) in terms of these new variables. Then, we multiply the first equation by \( \dot{\lambda}_e(1) \), integrate on the variable \( a \) on \([0, l]\) and integrate by parts using the boundary conditions; also we multiply the other resulting equations respectively by \( \dot{\lambda}_e(2), \dot{\lambda}_e(3) \) and \( \dot{\lambda}_e(4) \). By using then Young’s inequality, the conditions on the coefficients and the estimates for \((c^*, u^*, v^*, r^*_e) \) obtained in Proposition 9, by adding suitably the resulting differential inequalities and using Gronwall’s lemma, we obtain that

\[
\|\dot{\lambda}_e(1)\|_{L^\infty(0, T ; L^2(Q))} + \|\dot{\lambda}_e(2)\|_{L^2(Q)} + \|\dot{\lambda}_e(3)\|_{L^2(Q)} + \|\dot{\lambda}_e(4)\|_{L^2(Q)} \leq C.
\]

By using this in the corresponding differential equations, we then obtain \( \|\dot{\lambda}_e(1) + \lambda_e(1)\|_{L^2(Q)} \leq C \), \( \|\dot{\lambda}_e(2)\|_{L^2(Q)} \leq C \) and \( \|\dot{\lambda}_e(3)\|_{L^2(Q)} \leq C \). These results tell us that in fact \( \|\dot{\lambda}_e(2)\|_{L^2(Q)} + \|\dot{\lambda}_e(3)\|_{L^2(Q)} \leq C \). In all of these estimates, \( C \) is independent of \( \epsilon \).

Returning to the original variables, we obtain similar estimates for \( \lambda_e(1), \lambda_e(2), \lambda_e(3) \) and \( \lambda_e(4) \). From these estimates, we conclude that there are \( \lambda(1), \lambda(2), \lambda(3), \lambda(4) \) and subsequences (we do not relabel them) such that the convergences in (83) hold.

Now we are ready to

**Proof of Theorem 4.** By using the convergences in (70) and (83), we can pass the limit as \( \epsilon \to 0+ \) in (78)–(80) to obtain (13)–(15), which concludes the proof of the theorem.

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**References**


